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Several Complex Variables and Complex Manifolds II

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Preface.

Since we have already given a general outline of the aim and scope of these notes in the preface to *Several Complex Variables and Complex Manifolds I*, we shall do no more here than provide a brief description of the contents of this volume together with a few notes of guidance to the reader.

Chapter 5 of the present notes is devoted to calculus on complex manifolds. The first four sections cover basic linear algebra and calculus on a differential manifold. Most of the material in these sections should be familiar (though perhaps not the notation) and I would suggest reading through them quickly, referring back to them later, if necessary, for specific results and notation. The next three sections lead up to the construction of the $\bar{\partial}$-operator on an arbitrary complex manifold and also describe the $\bar{\partial}$-operator for holomorphic bundle valued forms. In section 8 we prove the Dolbeault-Grothendieck lemma and solve the Cousin problems for polydiscs in $\mathbb{C}^n$. In section 9 we show how holomorphic vector bundles naturally enter into the study of compact complex manifolds. We discuss, for example, the Euler sequence for projective space; the geometric genus; theta functions for complex tori. Finally in section 10, we discuss various pseudoconvexivity conditions for non-compact complex manifolds.

Chapter 6 is a self-contained introduction to the theory of sheaves in complex analysis. Section 1 is devoted to sheaves and presheaves with many examples. In section 2 we show how sheaf theory can be used to construct the envelope of holomorphy of a domain spread in $\mathbb{C}^n$. This section is not used elsewhere in the text and may be omitted at first reading. In section 3 we define sheaf cohomology using fine resolutions. After proving Leray's theorem, we go on to define Čech cohomology and prove that it is naturally isomorphic to cohomology computed using fine reaolutions. There are many important illustrations of sheaf cohomology arguments in this section. For example: The de Rham isomorphism between singular and de Rham cohomology; The Dolbeault isomorphism theorem; the first Chern class and classification of complex line bundles.

In Chapter 7 we prove a number of foundational results in the theory of complex manifolds. In section 1 we define coherence and prove Oka's theorem on the coherence of the sheaf of relations. In section 2 we prove Cartan's theorems A and B granted the exactness of the $\bar{\partial}$-sequence for locally free sheaves (A proof of this result will be included in the projected part III of these notes; proofs may also be found in Hörmander [1] or Vesentini [1]). In section 3 we prove the finiteness theorem of Cartan and Serre and in section 4 the finiteness theorem of Grauert. In section 5 we prove Serre's theorems A and B and give a number of applications. In section 6 we prove Grauert's vanishing theorem and, following Grauert, show how it may be used to prove the Kodaira embedding theorem.

Contents

CHAPTER 5. CALCULUS ON COMPLEX MANIFOLDS

Introduction.

In sections 1 and 2 we cover basic linear algebra and calculus on differential manifolds. We define the complexification of a real vector space in section 3 and in section 4 we develop the main results of complex linear algebra that we need in the sequel. After giving some general facts about complex and holomorphic vector bundles in section 5 and constructing the tangent and cotangent bundles of a complex manifold in section 6 we reach the heart of the chapter in section 7 with the construction of the $\bar{\partial}$-operator on an arbitrary complex manifold. In section 8 we prove the Dolbeault-Grothendieck Lemma and, with a little more effort, deduce that the Cousin I and II problems are always solvable on polydiscs. Section 9 is devoted to the discussion of a number of important examples on compact complex manifolds. Thus we construct the Euler sequence for projective space, prove some basic results about linear systems and their relationship with holomorphic line bundles and conclude with a discussion of theta functions and complex tori. Finally, in section 10, we define the concepts of q-pseudoconvexivity, q-completeness and weakly positive vector bundles.

§1. Review of linear algebra

In this section we shall review some elementary linear algebra that does not depend on the assumption of an inner product structure. We shall generally omit proofs (usually simple) referring the reader to any one of the many texts in linear algebra. All vector spaces in this section will be assumed real and finite dimensional.

Definition 5.1.1. A *graded vector space* $\mathbf{E}$ is a collection $\{E_0,E_1,\ldots\}$ of vector spaces indexed by the positive integers. We write $\mathbf{E} = \{E_0,E_1,\ldots\}$ and set $\mathbf{E}_n = E_n$. We call E_n the nth. component of $\mathbf{E}$. A *morphism* $\mathbf{T}: \mathbf{E} \longrightarrow \mathbf{F}$, of degree $r \geq 0$, of graded vector spaces is a collection of linear maps $T_i: E_i \to F_{i+r}$ indexed by the positive integers.

Examples.

1. Associated to the vector space E we may define the graded vector spaces $\otimes E$, ΛE and $\odot E$ whose nth. components are respectively $\otimes^n E$, $\Lambda^n E$ and $\odot^n E$ (nth. tensor, exterior and symmetric powers of E respectively).

2. A linear map T: E → F induces degree zero morphisms $\otimes T: \otimes E \longrightarrow \otimes F$, $\Lambda T: \Lambda E \longrightarrow \Lambda F$, $\odot T: \odot E \longrightarrow \odot F$ whose nth. components are $\otimes^n T$, $\Lambda^n T$ and $\odot^n T$ respectively.

Definition 5.1.2. Let $\mathbf{E}$ and $\mathbf{F}$ be graded vector spaces. Then

1. The *direct sum* of $\mathbf{E}$ and $\mathbf{F}$ is the graded vector space $\mathbf{E} \oplus \mathbf{F}$ whose nth. component is $E_n \oplus F_n$.

2. The *tensor product* of $\mathbf{E}$ and $\mathbf{F}$ is the graded vector space $\mathbf{E} \otimes \mathbf{F}$ whose nth. component is $\bigoplus_{r+s=n} E_r \otimes F_s$.

Definition 5.1.3. A *graded vector space algebra* consists of a graded vector space $\mathbf{E}$ together with a morphism of degree zero $\phi: \mathbf{E} \otimes \mathbf{E} \longrightarrow \mathbf{E}$, written $\phi(A \otimes B) = AB$, satisfying the following properties

1. The multiplication defined by ϕ is associative.

2. There exists a unit element in $\mathbf{E}$ for the multiplication.

The algebra is said to be *commutative* if $AB = (-1)^{pq}BA$, $A \in E_p$, $B \in E_q$.

Remark. The universal factorization property for the tensor product implies that we may equivalently suppose that ϕ is a bilinear map on $\mathbf{E} \times \mathbf{E}$.

Example 3. Given a vector space E, $\otimes E$, ΛE and $\odot E$ are graded vector space algebras with respective algebra operations of tensor product, exterior or wedge product and symmetric product. ΛE is commutative.

Proposition 5.1.4. Let E and F be vector spaces. Then $\Lambda E \otimes \Lambda F$ has the natural structure of a commutative graded algebra with wedge product defined by

$$(X \otimes Y) \wedge (X' \otimes Y') = (-1)^{pq}(X \wedge X') \otimes (Y \wedge Y'),$$

where $X \in \Lambda E$, $Y \in \Lambda^p F$, $X' \in \Lambda^q E$, $Y' \in \Lambda F$.

Proof. We remark only that in all results of this type it is enough to define the operation or map on a set of generators for the algebra. □

Before stating the next result we remark that we say a morphism of graded vector spaces is an *isomorphism* if it is invertible and of degree zero. An isomorphism of graded vector space algebras will, in addition, preserve the algebra structures.

Theorem 5.1.5. If E and F are vector spaces we have a canonical isomorphism of commutative graded algebras

$$\Lambda(E \oplus F) \approx \Lambda E \otimes \Lambda F \ .$$

In particular,

$$\Lambda(E \oplus F))_p \approx \bigoplus_{r+s=p} \Lambda^r E \otimes \Lambda^s F, \ p \geq 0.$$

Proof. For $r,s \geq 0$ we define $\mu_{r,s}\colon \Lambda^r E \otimes \Lambda^s F \to \Lambda(E \oplus F))_{r+s}$ by $\mu_{r,s}(e_1 \wedge \dots \wedge e_r \otimes f_1 \wedge \dots \wedge f_s) = (e_1 + 0) \wedge \dots \wedge (e_r + 0) \wedge (f_1 + 0) \wedge \dots \wedge (f_0 + 0)$, where $e_i \in E$, $1 \leq i \leq r$, $f_j \in F$, $1 \leq j \leq s$, and the wedge product on the right hand side of the above relation is taken in $\Lambda^{r+s}(E \oplus F)$. We leave it to the reader to verify that the maps $\mu_{r,s}$ define the required morphism. □

Given a vector space E it is possible to identify ΛE and $\odot E$ with graded subspaces (*not* subalgebras) of $\otimes E$ and we now indicate how to do this in the case of exterior powers of E. Fix a positive integer p and let S_p denote the symmetric group on p symbols. We define a representation $T\colon S_p \to GL(\otimes^p E)$ of S_p by

$$T(\sigma)(e_1 \otimes \dots \otimes e_p) = \operatorname{sign}(\sigma) e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(p)},$$

where $e_j \in E$, $1 \leq j \leq p$, $\sigma \in S_p$. Let $Alt_p(E)$ denote the fixed point set of the corresponding action of S_p on $\otimes^p E$. Elements of $Alt_p(E)$

are called *alternating tensors* of order p. The map $\wedge: \otimes^p E \to \Lambda^p E$ defined by mapping $e_1 \otimes \ldots \otimes e_p$ to $e_1 \wedge \ldots \wedge e_p$ induces a linear isomorphism of $\mathrm{Alt}_p(E)$ with $\Lambda^p E$. To see this observe that we have a projection map $\mathrm{Alt}: \otimes^p E \to \mathrm{Alt}_p(E)$ defined by

$$\mathrm{Alt}(e_1 \otimes \ldots \otimes e_p) = \frac{1}{p!} \sum_{\sigma \in S_p} T(\sigma)(e_1 \otimes \ldots \otimes e_p).$$

Clearly Kernel(Alt) = Kernel($\wedge$) and so $\mathrm{Alt}_p(E) \approx \Lambda^p E$.

Remark. Under the identification of $\Lambda^p E$ with $\mathrm{Alt}_p(E)$ given by this isomorphism we see that

$$e_1 \wedge \ldots \wedge e_p = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sign}(\sigma) e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(n)}.$$

Let $E' = L_{\mathbb{R}}(E, \mathbb{R})$ denote the dual space of E and $\langle\ ,\ \rangle: E \times E' \to \mathbb{R}$ denote the dual pairing between E and E'. For $p \geq 0$, we have dual pairings $\otimes^p E \times \otimes^p E' \to \mathbb{R}$, $\Lambda^p E \times \Lambda^p E' \to \mathbb{R}$ respectively defined by

$$\langle e_1 \otimes \ldots \otimes e_p, \phi_1 \otimes \ldots \otimes \phi_p \rangle = \prod_{i=1}^{p} \langle e_i, \phi_i \rangle$$

$$\langle e_1 \wedge \ldots \wedge e_p, \phi_1 \wedge \ldots \wedge \phi_p \rangle = \det[\langle e_i, \phi_j \rangle],$$

where $e_i \in E$, $\phi_i \in E'$, $1 \leq i \leq p$. Using these pairings we identify the dual spaces of $\otimes^p E$, $\Lambda^p E$ with $\otimes^p E'$, $\Lambda^p E'$ respectively. Notice that the dual pairing of $\otimes^p E$ with $\otimes^p E'$ induces a pairing of $\mathrm{Alt}_p(E)$ with $\mathrm{Alt}_p(E')$ and so a pairing of $\Lambda^p E$ with $\Lambda^p E'$. This pairing differs from the pairingwe have defined between $\Lambda^p E$ and $\Lambda^p E'$ by a factor p!

We leave the case of the dual pairing for the symmetric powers of E as an exercise.

For $p > 0$, we define the linear map $U: \Lambda^p E \to E \otimes \Lambda^{p-1} E$ by

$$U(e_1 \wedge \ldots \wedge e_p) = \sum_{j=1}^{p} (-1)^{j+1} e_j \otimes e_1 \wedge \ldots \wedge \widehat{e_j} \wedge \ldots \wedge e_p,$$

where $e_j \in E$, $1 \leq j \leq p$, and "$\widehat{\ }$" as usual denotes omission.

Properties of the map U.

1. The operator U is the transpose of $\wedge$, that is,

$$\langle UX, \phi \otimes \psi \rangle = \langle X, \phi \wedge \psi \rangle,$$

for all $X \in \Lambda^p E$, $\phi \in E'$, $\psi \in \Lambda^{p-1} E'$.

2. If $X \in \Lambda^p E$, $\wedge U(X) = pX$.

3. If $Z \in E \otimes \Lambda^p E$, then $U(\wedge Z) = Z - \wedge^{13}(I \otimes U)(Z)$, where $\wedge^{13}$ denotes the operation of wedging the first factor of $E \otimes E \otimes \Lambda^{p-1} E$ into the third factor.

4. If we iterate U, the map $U^p : \Lambda^q E \to \otimes^p E \otimes \Lambda^{q-p} E$ satisfies the relation $\wedge U^p = q!/(q-p)!\, I$, where $\wedge : \otimes^p E \otimes \Lambda^{q-p} E \to \Lambda^q E$ is just the operation of wedge product. In particular, if we identify $\Lambda^p E$ with $Alt_p(E)$, we see that $Alt\, U^p = p! I$, $p \geq 0$.

Properties 1 - 3 are easily proved by working on generators, property 4 may be proved by induction on p.

We conclude this section by describing the operation of *contraction* or *interior product* of tensors. First observe that since the dual pairing $E' \times E \to \mathbb{R}$ is bilinear it corresponds to a linear map $E' \otimes E \to \mathbb{R}$. This map is usually referred to as "trace" as if we use the natural identification of $E' \otimes E$ with $L(E,E)$ it corresponds to the operation of taking the trace of a linear map. Trace is the simplest example of a contraction operation and we shall now consider generalisations. With a view to later applications we work with tensor products of *exterior* powers of vector spaces. Actually this is no loss of generality since $\Lambda^1 E = E$, $\Lambda^1 E' = E'$.

Suppose $p \geq q \geq 0$. We define the linear map

$$\mathbf{C} : \Lambda^p E \otimes \Lambda^q E' \to \Lambda^{p-q} E$$

by requiring that

$$\langle \mathbf{C}(X \otimes \phi), \psi \rangle = \langle \phi \wedge \psi, X \rangle$$

for all $X \in \Lambda^p E$, $\psi \in \Lambda^{p-q}E'$, $\phi \in \Lambda^q E'$. We similarly define

$$\mathbf{C}: \Lambda^p E \otimes \Lambda^q E' \to \Lambda^{q-p}E'$$

in case $q \geq p$.

Given $X \in \Lambda^p E$ and $q \geq p$ we define

$$\mathbf{C}_X: \Lambda^q E' \to \Lambda^{q-p}E'$$

by $\mathbf{C}_X(\phi) = \mathbf{C}(X \otimes \phi)$, $\phi \in \Lambda^q E'$. We similarly define $\mathbf{C}_\phi$ for $\phi \in \Lambda^p E'$. We call $\mathbf{C}$ a *contraction* (operator) and $\mathbf{C}_X$ *contraction with X*.

Properties of the contraction operators $\mathbf{C}$, $\mathbf{C}_X$, $\mathbf{C}_\phi$.

1. For $p \geq 0$, the map $\mathbf{C}: \Lambda^p E \otimes \Lambda^p E' \to \mathbb{R}$ is equal to the dual pairing of $\Lambda^p E$ with $\Lambda^p E'$.

2. $\mathbf{C}_X$ is the transpose of the operator "wedge product with X". That is,

$$\langle \mathbf{C}_X \phi, Y\rangle = \langle \phi, X \wedge Y\rangle, \quad X \in \Lambda^p E,\ Y \in \Lambda^{s-p}E,\ \phi \in \Lambda^s E .$$

3. $\mathbf{C}_X \mathbf{C}_Y = \mathbf{C}_{Y\wedge X} = (-1)^{rs}\mathbf{C}_{X\wedge Y}$, $X \in \Lambda^r E$, $Y \in \Lambda^s E$.

4. For $X \in E$, $\mathbf{C}_X: \Lambda^p E' \to \Lambda^{p-1}E'$ is defined on generators by

$$\mathbf{C}_X(\phi_1 \wedge \dots \wedge \phi_p) = \sum_{j=1}^{p} (-1)^{j+1}\langle X,\phi_j\rangle \phi_1 \wedge \dots \wedge \widehat{\phi_j} \wedge \dots \wedge \phi_p .$$

We have similar properties holding for contraction with forms. We omit proofs of the above properties which all follow straightforwardly from the definition and standard properties of wedge products.

We now wish to define contractions between factors of an arbitrary finite tensor product of exterior powers of E and E'. Suppose that $V = \bigotimes_{i=1}^{n} V_i$, where each V_i is either $\Lambda^{p_i}E$ or $\Lambda^{p_i}E'$. Assume that the jth. and kth. factors are equal to $\Lambda^{p_j}E$ and $\Lambda^{p_k}E'$ respectively. Set $\tilde{V} = \bigotimes_{i=1}^{n} \tilde{V}_i$, where $V_i = V_i$ if $i \neq j,k$ and $\tilde{V}_j = \mathbb{R}$, $\tilde{V}_k = \Lambda^{p_k - p_j}E'$ if $p_j \leq p_k$ and $\tilde{V}_j = \Lambda^{p_j - p_k}E$, $\tilde{V}_k = \mathbb{R}$ if $p_j \geq p_k$ (Notice that $\tilde{V}$ is really a tensor product of $n-1$ factors but inclusion of the trivial factor for the present simplifies our indexing). The contraction between the jth. and kth. factors of V will be the linear map $\mathcal{C}_k^j: V \to \tilde{V}$ defined on generators by

$$\mathcal{C}_k^j(\ldots \otimes X_j \otimes \ldots \otimes \phi_k \otimes \ldots) = \ldots \otimes 1 \otimes \ldots \otimes \mathcal{C}(X_j \otimes \phi_k) \otimes \ldots, \quad p_j \le p_k$$

$$= \ldots \otimes \mathcal{C}(X_j \otimes \phi_k) \otimes \ldots \otimes 1 \otimes \ldots, \quad p_j \ge p_k .$$

Notice that our convention is that superscripts refer to factors which are exterior powers of E, subscripts to factors which are exterior powers of E'. The *product* $\mathcal{C}_k^j\mathcal{C}_m^l$ of the contractions $\mathcal{C}_k^j$ and $\mathcal{C}_m^l$ is defined if $\{l,j\} \cap \{l,m\} = \emptyset$ and is the simultaneous contraction of the jth. and kth. and l and mth. factors of V. Necessarily, $\mathcal{C}_k^j\mathcal{C}_m^l = \mathcal{C}_m^l\mathcal{C}_k^j$. The *composition* $\mathcal{C}_k^j.\mathcal{C}_m^l$ is defined to be the composite of the contraction $\mathcal{C}_m^l$ with the contraction $\mathcal{C}_k^j$, where $\mathcal{C}_k^j$ is a contraction of the space $\tilde{V}$ not V. We may also use brackets to describe compositions of contractions. Thus if $Z \in V$, $\mathcal{C}_k^j.\mathcal{C}_m^l(Z) = \mathcal{C}_k^j(\mathcal{C}_m^l(Z))$. Notice that $\tilde{V}$ has fewer factors than V and so, in general, $\mathcal{C}_k^j.\mathcal{C}_m^l \ne \mathcal{C}_k^j\mathcal{C}_m^l$.

Example 4. Let $p \ge r+s$. Then the following diagram of contractions commutes.

$$\begin{array}{ccc}
\Lambda^r E \otimes \Lambda^p E' \otimes \Lambda^s E & \xrightarrow{\ \mathcal{C}_2^3\ } & \Lambda^r E \otimes \Lambda^{p-s} E' \\
\downarrow \mathcal{C}_2^1 & & \downarrow \mathcal{C} \\
\Lambda^{p-r} E' \otimes \Lambda^s E & \xrightarrow{\ (-1)^{rs}\mathcal{C}\ } & \Lambda^{p-s-r} E'
\end{array} .$$

Indeed, let $X \in \Lambda^r E$, $Y \in \Lambda^s E$, $Z \in \Lambda^{p-r-s} E$, $\phi \in \Lambda^p E'$. Then

$$\langle \mathcal{C}(\mathcal{C}_2^3(X \otimes \phi \otimes Y)), Z\rangle = \langle \mathcal{C}_{X\wedge Y}\phi, Z\rangle$$

$$= (-1)^{rs}\langle \mathcal{C}_Y\mathcal{C}_X\phi, Z\rangle, \text{ Property 2 of contractions.}$$

$$= (-1)^{rs}\langle \mathcal{C}_2^1(\mathcal{C}(X \otimes \phi \otimes Y)), Z\rangle.$$

Since this is true for all $Z \in \Lambda^{p-r-s}E$, our assertion follows. In our notation above we have shown $\mathcal{C}.\mathcal{C}_2^3 = (-1)^{rs}\mathcal{C}.\mathcal{C}_2^1$.

We now list, without proof, some additional properties of contraction operators.

Properties of the contraction operators $\mathcal{C}$, $\mathcal{C}_X$, $\mathcal{C}_\phi$ continued.

5. Let $X \in \Lambda^p E$, $\phi \in \Lambda^q E'$ and suppose $q \geq p$. Then

$$\mathcal{C}_X\phi = \frac{1}{(q-p)!p!}\mathcal{C}^1_{p+1} \cdots \mathcal{C}^p_{2p}(U^p X \otimes U^p \phi).$$

In particular, if $X \in E$,

$$\mathcal{C}_X\phi = \mathcal{C}^1_2(X \otimes U\phi).$$

6. Let $X \in \Lambda^p E$, $\phi \in \Lambda^{p+1}E'$ then

$$\mathcal{C}_X\phi = (-1)^p \mathcal{C}^3_1(\phi \otimes UX).$$

7. Let $X \in E$, $\phi \in \Lambda^p E'$, $\psi \in \Lambda^q E'$. Then

$$\mathcal{C}_X(\phi \wedge \psi) = \mathcal{C}_X(\phi) \wedge \psi + (-1)^p \phi \wedge \mathcal{C}_X\psi.$$

8. Let $p \geq q > 0$ and define

$$\mathcal{C}_1 : \Lambda^p E \otimes \Lambda^q E' \to \Lambda^{p-1}E \otimes \Lambda^{q-1}E'$$

by

$$\mathcal{C}_1(X \otimes \phi) = \mathcal{C}^1_3(UX \otimes U\phi), \; X \in \Lambda^p E, \; \phi \in \Lambda^q E'.$$

Then $\mathcal{C} = (\mathcal{C}_1)^p/p!: \Lambda^p E \otimes \Lambda^q E' \to \Lambda^{p-q}E$.

We end this section with an example giving another characterization of trace.

Example 5. Let $A \in L(E,E)$ and $\dim(E) = n$. Then the map $\tilde{A}: \Lambda^n E \to \Lambda^n E$, defined on generators by

$$\tilde{A}(Z_1 \wedge \ldots \wedge Z_n) = \sum_{i=1}^{n} Z_1 \wedge \ldots \wedge A(Z_i) \wedge \ldots \wedge Z_n ,$$

is equal to scalar multiplication by -trace(A). Working on generators in $L(E,E)$, let $A = \phi \otimes X \in E' \otimes E \approx L(E,E)$ and $Z \in \Lambda^n E$. Then $\tilde{A}(Z) = X \wedge \mathcal{C}_\phi Z = -(\mathcal{C}_\phi X)Z$, Property 7 of contractions. But $\mathcal{C}_\phi X$ is just trace(A).

Exercises.

1. Verify that $\otimes^p E'$ is naturally isomorphic to the space $L^p(E;\mathbb{R})$ of p-linear real valued maps on E. Show also that $\wedge^p E'$ and $\vee^p E'$ are naturally isomorphic to the spaces of p-linear alternating and symmetric real valued maps on E respectively.

2. Let $p \le q$. Show that $\mathbf{C}$: $\wedge^p E \otimes \wedge^q E' \to \wedge^{q-p} E'$ is defined in terms of generators by the formula

$$\mathbf{C}(X_1 \wedge \dots \wedge X_p \otimes \phi_1 \wedge \dots \wedge \phi_q) = \sum_{I,J} \text{sign}(I,J) \langle X_1, \phi_{i_1} \rangle \dots \langle X_p, \phi_{i_p} \rangle \phi_{j_1} \wedge \dots \wedge \phi_{j_{q-p}},$$

where the sum is taken over all p-tuples $I = (i_1, \dots, i_p)$ satisfying $1 \le i_1, \dots, i_p \le q$ and (q-p)-tuples $J = (j_1, \dots, j_{q-p})$ satisfying $1 \le j_1 < \dots < j_{q-p} \le q$, subject to the condition that $\{i_1, \dots, j_{q-p}\}$ is a permutation of $\{1, \dots, q\}$. Sign(I,J) denotes the signature of this permutation.

3. Work out $\mathbf{C}_1$: $\wedge^2 E \otimes \wedge^2 E' \to E \otimes E'$ in terms of generators and verify that $(\mathbf{C}_1)^2 = 2\mathbf{C}$.

4. Generalise the example at the end of the section to find other invariants of the map A.

5. Let $0 \to E \xrightarrow{A} F \xrightarrow{B} G \to 0$ be a short exact sequence of vector spaces and linear maps. Suppose that the dimensions of E, F, G are p, q, r respectively. Prove that there exists a canonical isomorphism $\wedge^q F \approx \wedge^p E \otimes \wedge^r G$ (Hint: Prove that there exists a natural monomorphism $k: \wedge^r G' \otimes \wedge^q F \to \wedge^{q-r} F$ defined by $\langle k(\phi \otimes X), \psi \rangle = \langle \wedge^r B'(\phi) \wedge \psi, X \rangle$, $\phi \in \wedge^r G'$, $X \in \wedge^q F$, $\psi \in \wedge^{q-r} F'$. Show that image(k) = Image($\wedge^p A$)). More generally, show that for $n \ge 1$ we have a natural exact sequence

$$0 \to \wedge^n E \to \wedge^n F \to \wedge^{n-1} E \otimes F \to 0$$

6. Let dim(E) = n and $\phi \in E'$, $\phi \ne 0$. Show that the sequence

$$0 \to \wedge^n E \xrightarrow{\mathbf{C}_\phi} \wedge^{n-1} E \to \dots \xrightarrow{\mathbf{C}_\phi} \wedge^1 E \xrightarrow{\mathbf{C}_\phi} \mathbb{R} \to 0$$

is exact (Hint: Choose an orthonormal basis for E such that ϕ is an element of the dual orthonormal basis for E').

7. Continuing with the assumptions of question 6, verify that

a) $L(\Lambda^p E, \mathbb{R}) \approx \Lambda^p E' \approx \Lambda^n E' \otimes \Lambda^{n-p} E$, $0 \le p \le n$.

b) The diagram

$$\begin{array}{ccc} L(\Lambda^p E, \mathbb{R}) & \approx & \Lambda^n E' \otimes \Lambda^{n-p} E \\ \downarrow (C_\phi)' & & \downarrow I \otimes C_\phi \\ L(\Lambda^{p+1} E, \mathbb{R}) & \approx & \Lambda^n E' \otimes \Lambda^{n-p-1} E \end{array}$$

commutes, $0 \le p \le n$.

(Part b) amounts to saying that the sequence of question 6 is "self-dual").

§2. Calculus on differential manifolds.

In this section we review that portion of calculus on manifolds that does not depend on a choice of Riemannian metric. Proofs and further details may be found in Kobayashi and Nomizu [1] and Abraham and Marsden [1]. For the theory of vector bundles we refer in addition to the books by Husmoller [1] and Lang [1] and to the basic theory outlined in §5 of Chapter 1.

Throughout this section M will denote a C^∞ differential manifold of dimension m. For this section only, $C^r(M)$ will denote the space of *real* valued C^r functions on M, $0 \le r \le \infty$.

Let E be a smooth (that is, C^∞) vector bundle on M. For $r \ge 0$ we let $C^r(E)$ denote the vector space of C^r sections of E and $C^r_c(E)$ denote the space of C^r sections with compact support.

Notation and examples (see also §5 of Chapter 1).

1. We let $\underline{\mathbb{R}} = M \times \mathbb{R}$ denote the trivial real line bundle over M. Notice that $C^r(\underline{\mathbb{R}}) = C^r(M)$.

2. We let $\mathcal{T}M$ and $\mathcal{T}'M$ respectively denote the tangent and cotangent bundles of M.

As described in §5 of Chapter 1, we may form the dual bundle E' of a vector bundle E and tensor products of tensor, exterior and symmetric powers of E and E'. The contraction operations described in

§1 of this chapter all extend to these bundles and their sections. By way of example, there is a natural vector bundle map trace: $E \otimes E' \to \mathbb{R}$ defined by: $\mathrm{trace}(e_x \otimes \phi_x) = \langle e_x, \phi_x \rangle$, $e_x \in E_x$, $\phi_x \in E'_x$, $x \in M$. If S and ϕ are C^r sections of E and E' respectively, we may define the C^r section $\mathrm{trace}(S \otimes \phi)$ of $\underline{\mathbb{R}}$ by: $\mathrm{trace}(S \otimes \phi)(x) = \mathrm{trace}(S(x) \otimes \phi(x))$. In the sequel we use the same notation for contractions on vector bundles and their sections that we developed in the previous section for vector spaces.

Differential forms. A section of the bundle $\Lambda^p \mathcal{T}' M$ is called a *differential p-form* (on M). For $p \geq 0$ we have the operation $d: C^\infty(\Lambda^p \mathcal{T}' M) \to C^\infty(\Lambda^{p+1} \mathcal{T}' M)$ of *exterior differentiation* and the corresponding sequence

$$C^\infty(M) \xrightarrow{d} C^\infty(\mathcal{T}' M) \xrightarrow{d} \dots \xrightarrow{d} C^\infty(\Lambda^p \mathcal{T}' M) \xrightarrow{d} \dots .$$

Properties of exterior differentiation.

1. d is $\mathbb{R}$-linear.

2. $d^2 = 0$.

3. $d(\phi \wedge \zeta) = d\phi \wedge \zeta + (-1)^p \phi \wedge d\zeta$, $\phi \in C^\infty(\Lambda^p \mathcal{T}' M)$, $\zeta \in C^\infty(\Lambda^q \mathcal{T}' M)$.

4. If $f \in C^\infty(M)$, $x \in M$, then $df(x) = T_x f \in L(T_x M, \mathbb{R}) = \mathcal{T}'_x M$.

5. If $f: M \to N$ is C^∞ and $\phi \in C^\infty(\Lambda^p \mathcal{T}' N)$, then $d(f^*\phi) = f^* d\phi$ (f^* denotes the operation of pull-back of differential forms induced by f).

6. In local coordinates, if $\phi = \sum_{1 \leq i_1 < \dots < i_p \leq m} \phi_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$, where the coefficients $\phi_{i_1 \dots i_p}$ are C^∞ functions, then

$$d\phi = \sum_{j=1}^{m} \sum_{1 \leq i_1 < \dots < i_p \leq m} \partial \phi_{i_1 \dots i_p} / \partial x_j \, dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

We say that a p-form ϕ is *closed* if $d\phi = 0$ and that it is *exact* if, in addition, there exists a $(p-1)$-form ζ such that $d\zeta = \phi$. A closed form need not be exact though by Poincaré's lemma it is always *locally exact*. That is, if ϕ is an exact p-form on M and $x \in M$ we may find an open neighbourhood U of x (typically contractible) and a $(p-1)$-form ζ defined on U such that $d\zeta = \phi | U$.

The bundles $\Lambda^p\mathcal{T}^*M$ are zero bundles for $p > m$ and $\Lambda^m\mathcal{T}^*M$ is a real line bundle. We say that M is *orientable* if $\Lambda^m\mathcal{T}^*M$ is isomorphic to the trivial line bundle $\underline{\mathbb{R}}$. If M is orientable we have a natural $\mathbb{R}$-linear map $\int_M: C_c^0(\Lambda^m\mathcal{T}^*M) \to \mathbb{R}$ called *integration*. If M has boundary ∂M (necessarily orientable) then we have *Stokes' theorem:*

$$\int_{\partial M}\phi = \int_M d\phi, \ \phi \in C_c^1(\Lambda^{m-1}\mathcal{T}^*M) .$$

For $p \geq 0$, we define the pth. *de Rham cohomology group*, $H_{DR}^p(M,\mathbb{R})$, of M to be the quotient vector space

$$H_{DR}^p(M,\mathbb{R}) = (\text{Ker } d: C^\infty(\Lambda^p\mathcal{T}^*M) \to C^\infty(\Lambda^{p+1}\mathcal{T}^*M))/dC^\infty(\Lambda^{p-1}\mathcal{T}^*M).$$

Clearly, $H_{DR}^p(M,\mathbb{R}) = 0$, $p > m$. It is true, though by no means trivial, that if M is compact the de Rham cohomology groups are all finite dimensional vector spaces. It is also true that integration defines a dual pairing between $H_{DR}^p(M,\mathbb{R})$ and the singular homology group $H_p(M,\mathbb{R})$, $p \geq 0$. As a consequence, $H_{DR}^p(M,\mathbb{R})$ is isomorphic to $H^p(M,\mathbb{R})$, $p \geq 0$. We give a proof of this isomorphism in Chapter 6, §3.

Vector fields and Lie derivatives. We say that a map $\delta: C^\infty(M) \to C^\infty(M)$ is a *derivation* if δ is $\mathbb{R}$-linear and

$$\delta(fg) = (\delta f)g + f(\delta g),$$

for all $f,g \in C^\infty(M)$. We denote the set of derivations of $C^\infty(M)$ by D(M). We have a natural map of $C^\infty(\mathcal{T}M)$ into D(M), denoted $X \longmapsto L_X$, defined by

$$L_Xf = \lrcorner_X df, \ f \in C^\infty(M).$$

L_Xf is called the *Lie derivative* of f with respect to X. It is a basic result that this map of $C^\infty(\mathcal{T}M)$ into D(M) is a bijection. This allows us to think of vector fields as (1st. order) linear partial differential operators on $C^\infty(M)$.

Given $X,Y \in C^\infty(\mathcal{T}M)$, the map $L_XL_Y - L_YL_X$ is a derivation of $C^\infty(M)$ and so there exists a vector field Z such that $L_XL_Y - L_YL_X = L_Z$. In the sequel we call Z the *Lie bracket* of X and Y and denote it by [X,Y].

Properties of the Lie bracket.

1. [,] is $\mathbb{R}$-bilinear.

2. $[X,Y] = -[Y,X]$, $X,Y \in C^{\infty}(\mathcal{T}M)$.

3. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ for all $X,Y,Z \in C^{\infty}(\mathcal{T}M)$ (Jacobi identity).

4. If $f: M \to N$ is a C^{∞} diffeomorphism then $[f_*X,f_*Y] = f_*[X,Y]$, for all $X,Y \in C^{\infty}(\mathcal{T}M)$ (f_* is the operation of push-forward of vector fields induced by f).

We may give an invariant definition of exterior differentiation in terms of Lie derivatives and brackets. Suppose that ϕ is a differential p-form and $X_0,\ldots,X_p$ are vector fields on M then

$$\langle d\phi, X_0 \wedge \ldots \wedge X_p\rangle = \sum_{i=0}^{p} (-1)^i L_{X_i}\langle\phi, X_0 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_p\rangle$$

$$+ \sum_{0\le i<j} (-1)^{i+1}\langle\phi,[X_i,X_j],X_0 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_p\rangle.$$

For each $X \in C^{\infty}(\mathcal{T}M)$ define $L_X: C^{\infty}(\mathcal{T}M) \to C^{\infty}(\mathcal{T}M)$ by $L_XY = [X,Y]$. We refer to L_XY as the *Lie derivative* of Y with respect to X. We now extend the operator L_X to C^{∞} sections of arbitrary finite tensor and exterior products of $\mathcal{T}M$ and $\mathcal{T}'M$. We first define L_X on sections of $\mathcal{T}'M$. Suppose $\phi \in C^{\infty}(\mathcal{T}'M)$ then $L_X\phi$ will be the differential 1-form characterised by

$$L_X\langle\phi,Y\rangle = \langle L_X\phi,Y\rangle + \langle\phi,L_XY\rangle, \quad \text{for all } Y \in C^{\infty}(\mathcal{T}M).$$

It is quite straightforward, using the basic properties of Lie derivatives, to check that this relation does indeed define $L_X\phi$ as a section of $\mathcal{T}'M$. We shall extend L_X to arbitrary tensor and exterior powers by requiring that it is a derivation. Thus, for $p \ge 0$, we define $L_X: C^{\infty}(\Lambda^p\mathcal{T}M) \to C^{\infty}(\Lambda^p\mathcal{T}M)$ by

$$L_X(X_1 \wedge \ldots \wedge X_p) = \sum_{i=1}^{p} X_1 \wedge \ldots \wedge L_XX_i \wedge \ldots \wedge X_p$$

and $L_X: C^\infty(\Lambda^p\mathcal{T}'M) \to C^\infty(\Lambda^p\mathcal{T}'M)$ by

$$L_X(\phi_1 \wedge \cdots \wedge \phi_p) = \sum_{i=1}^{p} \phi_1 \wedge \cdots \wedge L_X\phi_i \wedge \cdots \wedge \phi_p ,$$

where X_i and ϕ_i are C^∞ vector fields and 1-forms on M respectively. Suppose that V and W are vector bundles which are finite tensor products of tensor and exterior powers of $\mathcal{T}M$ and $\mathcal{T}'M$ and that we have defined L_X on C^∞ sections of V and W. We define $L_X: C^\infty(V \otimes W) \to C^\infty(V \otimes W)$ by

$$L_X(S \otimes T) = L_X S \otimes T + S \otimes L_X T, \; S \in C^\infty(V), \; T \in C^\infty(W).$$

Again it is not hard to verify that L_X is well defined. Since we have already defined L_X on sections of exterior powers of $\mathcal{T}M$ and $\mathcal{T}'M$ the above construction allows us to extend L_X to arbitrary finite tensor products of tensor and exterior powers of $\mathcal{T}M$ and $\mathcal{T}'M$.

A most important property of Lie derivatives is that they commute with contractions. Indeed this property is built into the definition of Lie differentiation on differential 1-forms for we have

$$L_X\mathbb{C}(\phi \otimes Y) = \mathbb{C}(L_X\phi \otimes Y) + \mathbb{C}(\phi \otimes L_X Y),$$

$\phi \in C^\infty(\mathcal{T}'M)$, $X,Y \in C^\infty(\mathcal{T}M)$. We leave it to the reader to verify that Lie derivatives commute with any of the contraction operations we have so far defined on tensor products of $\mathcal{T}M$ and $\mathcal{T}'M$.

Exercises.

1. Let $X \in C^\infty(\mathcal{T}M)$, $\phi \in C^\infty(\Lambda^p\mathcal{T}'M)$, $f \in C^\infty(M)$. Verify the identities

   i) $L_X\phi = d\mathbb{C}_X\phi + \mathbb{C}_X d\phi$

   ii) $L_{fX}\phi = fL_X\phi + df \wedge \mathbb{C}_X\phi.$

2. Let $A \in C^\infty(\mathcal{T}'M \otimes \mathcal{T}M)$. Define $\tilde{A}: C^\infty(\mathcal{T}M) \to C^\infty(\mathcal{T}M)$ by $A(X) = \mathbb{C}_2^1(X \otimes A)$. Show that we can extend $\tilde{A}$, as a derivation, to sections of any finite tensor product of tensor and exterior powers of $\mathcal{T}M$ and $\mathcal{T}'M$ in such a way that $\tilde{A}$ is the zero operator on $C^\infty(M)$ and commutes with contractions. Show further that if Ω is any operator defined on sections of tensor

products of $\mathcal{T}M$ and $\mathcal{T}'M$ that is a derivation and commutes with contractions then there exist unique $X \in C^{\infty}(\mathcal{T}M)$ and $A \in C^{\infty}(\mathcal{T}'M \otimes \mathcal{T}M)$ such that $\Omega = L_X + \tilde{A}$.

3. Verify the following local forms for the Lie derivative and Lie bracket

a) $L_X f = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x_i}$.

b) $[X,Y]^i = \sum_{j=1}^{n} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right)$.

§3. Complexification.

Throughout this section tensor products over $\mathbb{R}$ and $\mathbb{C}$ will be denoted by $\otimes_{\mathbb{R}}$ and $\otimes_{\mathbb{C}}$ respectively. We use similar notation for exterior and symmetric powers. Omission of a subscript will generally indicate a product over $\mathbb{C}$ unless the contrary is clearly indicated.

Definition 5.3.1. Let E be a real vector space. The *complexification* ${}_cE$ of E is the vector space $E \otimes_{\mathbb{R}} \mathbb{C}$.

Properties of complexification.

1. ${}_cE$ has the natural structure of a complex vector space with scalar multiplication defined by $c(e \otimes_{\mathbb{R}} d) = e \otimes_{\mathbb{R}} cd$, $e \in E$, $c,d \in \mathbb{C}$.

2. $\dim_{\mathbb{C}}({}_cE) = \dim_{\mathbb{R}} E$.

3. ${}_cE$ has a natural splitting $E_R \oplus E_I$ into real and imaginary parts defined by $E_R = \{e \otimes_{\mathbb{R}} 1 : e \in E\}$; $E_I = \{e \otimes_{\mathbb{R}} i : e \in E\}$.

4. The operation of complexification commutes with tensor, exterior and symmetric products. For example, if $p \geq 0$, ${}_c(\otimes^p_{\mathbb{R}} E) \approx \otimes^p({}_cE)$ (as complex vector spaces).

5. ${}_c(E') \approx L_{\mathbb{R}}(E,\mathbb{C})$. This isomorphism is defined by mapping $\phi \otimes_{\mathbb{R}} c$ to $c\phi$. In the sequel we set ${}_c(E') = {}_cE'$.

6. The dual pairing $E \times E' \to \mathbb{R}$ complexifies to the dual pairing ${}_cE \times {}_cE' \to \mathbb{C}$. On generators, this pairing is defined by $\langle e \otimes_{\mathbb{R}} c, \phi \otimes_{\mathbb{R}} d \rangle = cd\langle \phi, e \rangle$, $c,d \in \mathbb{C}$, $\phi \in E'$, $e \in E$. Using this dual

pairing we identify the dual spaces of $\otimes^p{}_cE$, $\Lambda^p{}_cE$ with $\otimes^p{}_cE'$, $\Lambda^p{}_cE'$ respectively. We should stress that we shall *always* use this dual pairing in the sequel.

One of the main reasons that we introduce complexification is so that we can define conjugation.

Definition 5.3.2. The map $S: {}_cE \to {}_cE$ defined by $S(e \otimes_{\mathbb{R}} c) = e \otimes_{\mathbb{R}} \bar{c}$, $e \in E$, $c \in \mathbb{C}$, is called *conjugation*. We usually write $S(X) = \bar{X}$, $X \in {}_cE$.

Properties of conjugation.

1. $Si = -iS$ (conjugation is a conjugate complex linear map).

2. Let $X \in {}_cE$. Then $X = \bar{X}$ iff $X \in E_R$: $X = -\bar{X}$ iff $X \in E_I$.

3. Conjugation commutes with the operations of tensor, exterior and symmetric products. For example, conjugation on $\Lambda^p{}_cE$ may be defined as Λ^pS or, equivalently, as conjugation on ${}_c(\Lambda^p_{\mathbb{R}}E)$. In particular, notice that if $X_1 \wedge \ldots \wedge X_p \in \Lambda^p{}_cE$, then $\bar{X}_1 \wedge \ldots \wedge \bar{X}_p = \overline{X_1 \wedge \ldots \wedge X_p} \in \Lambda^p{}_cE$.

4. Using the natural isomorphism between ${}_cE'$ and $L_{\mathbb{R}}(E,\mathbb{C})$, we may regard conjugation as conjugation of functions. That is, if $f \in {}_cE$, we define $\bar{f} \in {}_cE'$ by $\bar{f}(e) = \overline{f(e)}$, $e \in E$.

5. Conjugation commutes with the dual pairing ${}_cE \times {}_cE' \to \mathbb{C}$ and so for all $X \in {}_cE$, $\phi \in {}_cE'$ we have

$$\overline{\langle X,\phi\rangle} = \langle \bar{X},\bar{\phi}\rangle .$$

6. The conjugate of a map $A \in L_{\mathbb{C}}({}_cE,{}_cF)$ is given by $\bar{A} = S.A.S$.

Finally, we remark that if ϕ is an element of some finite tensor product of tensor and exterior powers of ${}_cE$, ${}_cE'$ then ϕ is said to be *real* if $\phi = \bar{\phi}$. By property 2 above this is clearly equivalent to the existence of a real tensor γ such that $\phi = \gamma \otimes_{\mathbb{R}} 1$.

Exercises.

1. Let $X \in {}_cE$. Show that $X + \bar{X}$ is real, $X - \bar{X}$ is imaginary (that is, a point of E_I).

2. Let $A \in L_{\mathbb{R}}(E,F)$ and ${}_cA = A \otimes_{\mathbb{R}} 1 \in L_{\mathbb{C}}({}_cE, {}_cF)$ denote the complexification of A. Show that $\det_{\mathbb{C}}({}_cA) = \det_{\mathbb{R}}(A)$.

3. Let $A \in L_{\mathbb{C}}({}_cE, {}_cF)$. Show that $\det_{\mathbb{C}}(\bar{A}) = \overline{\det_{\mathbb{C}}(A)}$.

§4. Complex linear algebra.

This section, which may be regarded as a synthesis of §§1,3, summarizes the main results of complex linear algebra that we need in the sequel. Again we defer any consideration of inner product structures.

Suppose that E is a complex vector space. We let E^* denote the complex dual space $L_{\mathbb{C}}(E,\mathbb{C})$ of E. The theory of contractions that we described in §1 immediately extends to (complex) tensor products of tensor, exterior and symmetric powers of E and E^*. We continue to use the notation developed in §1.

Definition 5.4.1. (See also §5, Chapter 1). Let E be a real vector space. An endomorphism J of E is said to define a *complex structure* on E if $J^2 = -I$.

If E has complex structure J then E may be given the structure of a complex vector space if we define $(a+ib)e = a+bJ(e)$, $a,b \in \mathbb{R}$, $e \in E$. In particular, $\dim_{\mathbb{R}} E$ must be even. Conversely, if E is a complex vector space we may define a complex structure J on E by $j(e) = ie$, $e \in E$.

Definition 5.4.2. If E is a vector space with complex structure J, we define $\bar{E}$, the *conjugate* of E, to be the vector space E with complex structure $-J$.

Remark. Suppose E is a complex vector space and that $X \in E$ has coordinates $(z_1,\ldots,z_n)$ relative to some (complex) basis B of E. Then B is also a baais of $\bar{E}$ and the coordinates of $X \in \bar{E}$ are $(\bar{z}_1,\ldots,\bar{z}_n)$. In this sense, taking the conjugate of a complex vector space corresponda to taking conjugates of the coordinates. Shortly, we shall use the device of complexification to give a "coordinate-free" version of this important operation. Indeed, much of the formalism we now develop is directed towards obtaining a satisfactory definition of conjugation for complex vector spaces that does not depend on a choice of coordinate system and can be extended to complex manifolds.

Properties of complex structures and conjugate spaces.

1. Let E, F be vector spaces with complex structures J_E, J_F respectively. A map $A \in L_{\mathbb{R}}(E,F)$ is complex linear iff $A.J_E = J_F.A$. The space $L_{\mathbb{C}}(E,F)$ has the natural complex structure J defined by $J(A) = A.J_E = J_F.A$. We may define two *distinct* complex structures J_1, J_2 on $L_{\mathbb{R}}(E,F)$ by $J_1(A) = A.J_E$; $J_2(A) = J_F.A$

From now on assume that E is a vector space with complex structure J.

2. The complex structure on E* is defined by $J(\phi) = \phi.J = i\phi$, $\phi \in E^*$.

3. $\bar{E}^* \approx \overline{E^*}$ as complex vector spaces. The isomorphism is defined by mapping $\phi \in \bar{E}^*$ $(=L(\bar{E},\mathbb{C}))$ to $\bar{\phi} \in E^*$, where $\bar{\phi}(e) = \overline{\phi(e)}$, $e \in E$. Notice that the complex structure on $\bar{E}^*$ is defined by $J(\phi) = \phi.(-J) = i\phi$. In particular, $\bar{E}^*$ is the space of conjugate complex linear maps on E.

4. The operation of taking the conjugate space commutes with the operations of taking dual, tensor, exterior and symmetric powers. For example $\overline{\otimes^p E} \approx \otimes^p \bar{E}$ as complex vector spaces.

As we described in §3, we may give ${}_cE$ the structure of a complex vector space. Clearly the complexification of J, which we continue to denote by J, also defines a complex structure on ${}_cE$. As we shall soon see these two complex structures on ${}_cE$ are different. In the sequel, we always give ${}_cE$ the complex vector space structure induced from $\mathbb{C}$ and never that induced from J.

Define $P, \bar{P}: {}_cE \to {}_cE$ by

$$P = \tfrac{1}{2}(I - iJ), \quad \bar{P} = \tfrac{1}{2}(I + iJ).$$

Clearly, P, $\bar{P}$ are complementary projections: $P^2 = P$, $\bar{P}^2 = \bar{P}$ and $P + \bar{P} = I$. We set $\mathcal{E} = P({}_cE)$, $\bar{\mathcal{E}} = \bar{P}({}_cE)$. Then $\mathcal{E}$, $\bar{\mathcal{E}}$ are complementary complex subspaces of ${}_cE$ and so ${}_cE = \mathcal{E} \oplus \bar{\mathcal{E}}$. Observe that $J|\mathcal{E} = +i$,

$J|\bar{E} = -i$ and $S(E) = \bar{E}$ (hence the notation). We have natural isomorphisms $j: E \to E$; $\bar{j}: \bar{E} \to \bar{E}$ defined by

$$j(e) = \tfrac{1}{2}(e \otimes 1 - Je \otimes i); \quad \bar{j}(e) = \tfrac{1}{2}(e \otimes 1 + Je \otimes i) .$$

Hence we see that ${}_cE \approx E \oplus \bar{E}$. Notice that if we regard this isomorphism as an identification then E, $\bar{E}$ become complementary complex subspaces of ${}_cE$ in such a way that the complex structure on ${}_cE$ restricts to the complex structure J on E and the conjugate complex structure $-J$ on $\bar{E}$.

Let us now examine how conjugation fits into this framework. We have the commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{\ j\ } & E \oplus \bar{E} & = & {}_cE \\ \big\downarrow I & & \big\downarrow S & & \\ E & \xrightarrow{\ \bar{j}\ } & E \oplus \bar{E} & = & {}_cE \end{array}$$

We see that the identity map $I: E \to \bar{E}$, which is associated to taking conjugates of coordinates, corresponds to the invariantly defined operation of conjugation on ${}_cE$.

Properties of complexified linear maps.

1. Suppose E, F are complex vector spaces and $A \in L_{\mathbb{R}}(E,F)$. Let ${}_cA = A \otimes_{\mathbb{R}} 1 \in L_{\mathbb{C}}({}_cE, {}_cF)$ denote the complexification of A. We say that ${}_cA$ respects the splittings $E \oplus \bar{E}$, $F \oplus \bar{F}$ of ${}_cE$, ${}_cF$ (or just *splits*) if ${}_cA = A_1 \oplus A_2$, where $A_1: E \to F$, $A_2: \bar{E} \to \bar{F}$. We have the useful result that ${}_cA$ splits iff $A \in L_{\mathbb{C}}(E,F)$. Moreover, if ${}_cA$ splits, $A_2 = \bar{A}_1$. In the sequel, if $A \in L_{\mathbb{C}}(E,F)$ we write ${}_cA = A \oplus \bar{A}$ and observe that the following diagrams commute

$$\begin{array}{ccc} E & \xrightarrow{\ j\ } & E \oplus \bar{E} \\ \big\downarrow A & & \big\downarrow {}_cA = A \oplus \bar{A} \\ F & \xrightarrow{\ \bar{j}\ } & F \oplus \bar{F} \end{array} \qquad \begin{array}{ccc} \bar{E} & \xrightarrow{\ \bar{j}\ } & E \oplus \bar{E} \\ \big\downarrow \bar{A} & & \big\downarrow {}_cA = A \oplus \bar{A} \\ \bar{F} & \xrightarrow{\ \bar{j}\ } & F \oplus \bar{F} \end{array} .$$

The map $\bar{A}: \bar{E} \to \bar{F}$ is defined to be equal to A on the underlying real vector spaces of $\bar{E}$ and $\bar{F}$.

2. If $A \in L_{\mathbb{R}}(E,F)$, then $\det_{\mathbb{C}}({}_cA) = \det_{\mathbb{R}}(A)$ (Exercise 2, §3). Hence if $A \in L_{\mathbb{C}}(E,F)$, $\det_{\mathbb{R}}(A) = \det_{\mathbb{C}}(A \oplus \bar{A}) = \det_{\mathbb{C}} A \det_{\mathbb{C}} \bar{A} = \det_{\mathbb{C}} A \overline{\det_{\mathbb{C}} A} = |\det_{\mathbb{C}} A|^2 = |\det_{\mathbb{C}} A|^2$, since $A = jAj^{-1}$. In particular, the real determinant of a complex linear map is always positive (this implies, for example, the orientability of complex manifolds).

Next we turn to dual spaces. Recall from §3 that ${}_cE'$ may be identified with $L_{\mathbb{R}}(E,\mathbb{C})$. As described above we have complementary projection maps $P,\bar{P}$: ${}_cE' \to {}_cE'$ defined by

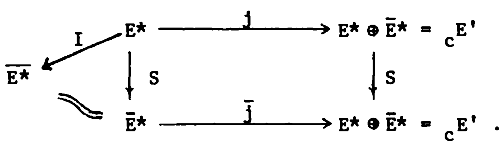
$$P(\phi) = \tfrac{1}{2}(\phi - i\phi.J), \quad \bar{P}(\phi) = \tfrac{1}{2}(\phi + i\phi.J), \quad \phi \in {}_cE' .$$

Observe that for all $\phi \in {}_cE'$, $P(\phi) \in E^*$, $\bar{P}(\phi) \in \bar{E}^*$. Hence we have the splitting

$${}_cE' = E^* \oplus \bar{E}^* .$$

This splitting amounts to saying that every complex valued $\mathbb{R}$-linear form on E has a unique decomposition as a sum of a complex linear and conjugate complex linear form.

Let us see how conjugation fits into this picture. The conjugation map $S: E^* \to \bar{E}^*$, defined by taking conjugates of linear forms, is the restriction to E^* of the conjugate operator $S: {}_cE' \to {}_cE'$ defined in §3. Hence we have the commutative diagram



The maps j, $\bar{j}$ are just inclusion maps. Notice that the conjugation map $S: E^* \to \bar{E}^*$ factors through $\overline{E^*}$ and compare with the previous diagram that we gave for conjugation on ${}_cE$.

As described in §3, the dual pairing $E \times E' \to \mathbb{R}$ complexifiea to the dual pairing $\langle\ ,\ \rangle: {}_cE \times {}_cE' \to \mathbb{C}$. We now investigate how this pairing behaves with respect to the factors $\mathcal{E}$, $\bar{\mathcal{E}}$ of ${}_cE$ and E^*, $\bar{E}^*$ of ${}_cE'$.

Proposition 5.4.3. The dual pairing $\langle\ ,\ \rangle\colon {}_cE \times {}_cE' \to \mathbb{C}$ induces

1. The zero pairing between E and $\bar{E}^*$.

$\bar{1}$. The zero pairing between $\bar{E}$ and E^*.

2. A dual pairing of E and E^*.

$\bar{2}$. A dual pairing of $\bar{E}$ and $\bar{E}^*$.

Moreover, the dual pairings between E, E^* and $\bar{E}$, $\bar{E}^*$ are given explicitly by

$$\langle j(e),\phi\rangle = \phi(e),\quad e \in E,\ \phi \in E^*$$

$$\langle \bar{j}(e),\psi\rangle = \psi(e),\quad e \in E,\ \psi \in \bar{E}^*.$$

Proof. It is enough to verify statements 1, 2 as $\bar{1}$, $\bar{2}$ follow by conjugation. Suppose $X \in E$, $\phi \in \bar{E}^*$. Then there exist $e \in E$, $\zeta \in E'$ such that $X = j(e) = \frac{1}{2}(e \otimes 1 - Je \otimes i)$, $\phi = \bar{j}(e) = \frac{1}{2}(\zeta \otimes 1 + \zeta.J \otimes i)$. Now $\langle\phi,X\rangle = \frac{1}{4}(\langle\zeta,e\rangle - i^2\langle\zeta,J^2e\rangle + i\langle\zeta,Je\rangle - i\langle\zeta,Je\rangle) = 0$, proving 1. If instead $\phi \in E^*$, there exists $\zeta \in E'$ such that $\phi = \frac{1}{2}(\zeta \otimes 1 - \zeta.J \otimes i)$ and computing we find that

$$\langle\phi,X\rangle = \tfrac{1}{2}(\zeta(e) - i\zeta(Je)) = \phi(e)\ . \qquad \square$$

We now use Proposition 5.4.3 to examine how complex linear maps and their duals behave under complexification and conjugation. If $A\colon E \to F$ is $\mathbb{C}$-linear, we have induced $\mathbb{C}$-linear maps $\bar{A}\colon \bar{E} \to \bar{F}$, $A^*\colon F^* \to E^*$, $\bar{A}^*\colon \bar{F}^* \to \bar{E}^*$ defined by

1. $\bar{A}(e) = A(e)$, $e \in \bar{E}$ (where, as real vector spaces, $\bar{E} = E$).

2. $\langle A^*(f^*),e\rangle = f^*(A(e))$, $e \in E$, $f^* \in F^*$.

3. $\langle \bar{A}^*(\bar{f}^*),e\rangle = \bar{f}^*(A(e))$, $e \in E$, $\bar{f}^* \in \bar{F}^*$.

Since A is assumed $\mathbb{C}$-linear, ${}_cA = A \oplus \bar{A}$, ${}_cA' = A^* \oplus \bar{A}^*$ and we have, by Proposition 5.4.3, the following alternative characterizations of $\bar{A}$, A^*, $\bar{A}^*$.

1.' $\bar{A} = \bar{j}^{-1}A\bar{j}$.

2.' $\langle A^*(f^*),e\rangle = \langle A(e),f^*\rangle$, $e \in E$, $f^* \in F^*$.

3.' $\langle \bar{A}^*(\bar{f}^*),\bar{e}\rangle = \langle \bar{A}(\bar{e}),\bar{f}^*\rangle$, $\bar{e} \in \bar{E}$, $\bar{f}^* \in \bar{F}^*$.

The pairings here are all induced from the dual pairing of E and E'. Observe how the definitions are now much more natural. In particular, 3' is just the conjugate of 2'. This naturality, that allows us to commute conjugation with other operations such as contraction or tensor product, is one of the main advantages of working with the complexifications of E and E'.

Next we look at the exterior algebras of $_cE$ and $_cE'$. By Theorem 5.1.5 we have the canonical isomorphisms

$$\mu: \Lambda^p{}_cE \to \bigoplus_{r+s=p} \Lambda^r E \otimes \Lambda^s \bar{E}$$

$$\mu: \Lambda^p{}_cE' \to \bigoplus_{r+s=p} \Lambda^r E^* \otimes \Lambda^s \bar{E}^* .$$

We set $\Lambda^{r,s}(E) = \mu^{-1}(\Lambda^r E \otimes \Lambda^s \bar{E})$: $\Lambda^{r,s}(E') = \mu^{-1}(\Lambda^r E^* \otimes \Lambda^s \bar{E}^*)$, $r,s \geq 0$. For $p \geq 0$ we therefore have the direct sum decomposition

$$\Lambda^p{}_cE = \bigoplus_{r+s=p} \Lambda^{r,s}(E); \quad \Lambda^p{}_cE' = \bigoplus_{r+s=p} \Lambda^{r,s}(E').$$

An element of $\Lambda^{r,s}(E)$ (resp. $\Lambda^{r,s}(E')$) is called a *complex (r,s)-vector* (resp. a *complex (r,s)-form*).

Properties of the exterior algebras of $_cE$ and $_cE'$.

For properties 1 and 2 below we suppose $\dim_{\mathbb{C}}(E) = m$.

1. $\Lambda^{r,s}(E) = 0$ if either $r > m$ or $s > m$. Similarly for forms.

2. $\Lambda^{2m}{}_cE = \Lambda^{m,m}(E)$. Similarly for forms.

3. $\overline{\Lambda^{r,s}(E)} = \Lambda^{s,r}(E)$, $r,s \geq 0$. A necessary condition for a complex (r,s)-vector to be real is $r = s$. Similarly for forms.

4. If $X \in \Lambda^{r,s}(E)$, $Y \in \Lambda^{u,v}(E)$ then $X \wedge Y \in \Lambda^{r+u,s+v}(E)$, $r,s,u,v \geq 0$. Similarly for forms.

5. The dual pairing $\bigwedge^p_c E \times \bigwedge^p_c E' \to \mathbb{C}$ restricts to a dual pairing $\bigwedge^{r,s}(E) \times \bigwedge^{r,s}(E') \to \mathbb{C}$ which is given on generators by

$$\langle X_1 \wedge \ldots \wedge X_r \wedge Y_{\bar{1}} \wedge \ldots \wedge Y_{\bar{s}}, \phi_1 \wedge \ldots \wedge \phi_r \wedge \psi_{\bar{1}} \wedge \ldots \wedge \psi_{\bar{s}}\rangle$$

$$= \langle X_1 \wedge \ldots \wedge X_r, \phi_1 \wedge \ldots \wedge \phi_r\rangle\langle Y_{\bar{1}} \wedge \ldots \wedge Y_{\bar{s}}, \psi_{\bar{1}} \wedge \ldots \wedge \psi_{\bar{s}}\rangle,$$

where $X_j \in E$, $Y_{\bar{i}} \in \bar{E}$, $\phi_j \in E^*$, $\psi_{\bar{i}} \in \bar{E}^*$, $1 \leq j \leq r$, $1 \leq i \leq s$. If $r+s = u+v$ but $r \neq u$, then the induced pairing $\bigwedge^{r,s}(E) \times \bigwedge^{u,v}(E') \to \mathbb{C}$ is always zero (both properties follow from Proposition 5.4.3).

6. If $A \in L_{\mathbb{C}}(E,F)$, then $\bigwedge^p_c A$: $\bigwedge^p_c E \to \bigwedge^p_c F$ induces maps $\bigwedge^{r,s}(A)$: $\bigwedge^{r,s}(E) \to \bigwedge^{r,s}(F)$, $r+s = p$. Similarly for forms.

7. If $r \geq u$, $s \geq v$, we have the contraction operation

$$C\colon \bigwedge^{r,s}(E) \otimes \bigwedge^{u,v}(E') \to \bigwedge^{r-u,s-v}(E)$$

obtained as the restriction of the contraction berween $\bigwedge^{r+s}{}_c E$ and $\bigwedge^{u+v}{}_c E$. Similar remarks hold for the other contraction operations defined in §1.

We conclude this section by looking at bases for the spaces we have been considering.

Suppose $B = \{e_1,\ldots,e_m\}$ is a complex basis for E. Then B is a complex basis for $\bar{E}$ and $B_R = \{e_1, Je_1, \ldots, e_m, Je_m\}$ is a real basis for E. The dual real basis B'_R for E' is given by

$$B'_R = \{e'_1, (Je_1)', \ldots, e'_m, (Je_m)'\}$$

$$= \{e'_1, -Je'_1, \ldots, e'_m, -Je'_m\}, \text{ since } (Je_j)' = -Je'_j.$$

We now define bases $\mathcal{B}$, $\bar{\mathcal{B}}$, B^*, $\bar{B}^*$ for E, $\bar{E}$, E^*, $\bar{E}^*$ respectively. Set

$$\mathcal{B} = \{f_j \in E\colon f_j = j(e_j),\ 1 \leq j \leq m\}.$$

$$\mathcal{B} = \{f_{\bar{j}} \in \bar{E}\colon f_{\bar{j}} = \bar{j}(e_j),\ 1 \leq j \leq m\}$$

$$B^* = \{f_j^* \in E^*: f_j^* = e_j' - i(Je_j)' = e_j' + iJe_j', \ 1 \le j \le m\}$$

$$\bar{B}^* = \{f_{\bar{j}}^* \in \bar{E}^*: f_{\bar{j}}^* = e_j' + i(Je_j)' = e_j' - iJe_j', \ 1 \le j \le m\}.$$

Notice that $\bar{f}_j = f_{\bar{j}}$, $\overline{f_j^*} = f_{\bar{j}}^*$, $1 \le j \le m$. Hence our notation for the bases $\mathcal{B}$, $\bar{\mathcal{B}}$ and B^*, $\bar{B}^*$ (such pairs of bases are called *self-conjugate*). We see also that $\mathcal{B}$ and B^* are dual bases (for the pairing $\langle\ ,\ \rangle$) since

$$\langle f_j, f_j^* \rangle = \tfrac{1}{2}\langle e_j - iJe_j, e_j' - i(Je_j)' \rangle = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

$$\langle f_j, f_k^* \rangle = 0, \ j \ne k.$$

Similarly, $\bar{\mathcal{B}}$ and $\bar{B}^*$ are dual bases.

With respect to the self-conjugate basis that we have constructed on ${}_cE$, every $X \in {}_cE$ may be written uniquely in the form

$$X = \sum_{j=1}^{m} z^j f_j + \sum_{j=1}^{m} z^{\bar{j}} f_{\bar{j}}$$

where z^j, $z^{\bar{j}} \in \mathbb{C}$. The coordinates $(z^1,\ldots,z^m,z^{\bar{1}},\ldots,z^{\bar{m}})$ are called *self-conjugate* coordinates on ${}_cE$.

Suppose that F is another complex vector space with complex basis C and associated bases $\mathcal{C}$, $\bar{\mathcal{C}}$, C^*, $\bar{C}^*$ as described above for the basis B of E. Let $A \in L_{\mathbb{C}}(E,F)$ have matrix $[a_{ij}]$ with respect to the bases B and C. Then

$$[\bar{A}] = [\bar{a}_{ij}];\ [A^*] = [a_{ji}];\ [\bar{A}^*] = [\bar{a}_{ji}];[\mathcal{A}] = [a_{ij}];\ [\bar{\mathcal{A}}] = [\bar{a}_{ij}],$$

where the matrices are computed relative to the appropriate bases associated to B and C.

Finally suppose that $X \in \Lambda^{r,s}(E)$, $\phi \in \Lambda^{r,s}(E')$. In coordinates we may write X and ϕ uniquely in the form

$$X = \sum_{I,J} X^{I\bar{J}} f_I f_{\bar{J}}\ ; \quad \phi = \sum_{I,J} \phi_{I\bar{J}} f_I^* f_{\bar{J}}^*\ ,$$

where the summations are over all r-tuples $I = (i_1,\ldots,i_r)$ satisfying $1 \le i_1 < \ldots < i_r \le m$ and s-tuples $J = (j_1,\ldots,j_s)$ satisfying

$1 \le j_1 < \ldots < j_s \le m$ and we have used the abbreviated notations $f_I f_{\bar{J}}$ and $f_I^* f_{\bar{J}}^*$ for $f_{i_1} \wedge \ldots \wedge f_{i_r} \wedge f_{\bar{j}_1} \wedge \ldots \wedge f_{\bar{j}_s}$ and $f_{i_1}^* \wedge \ldots \wedge f_{i_r}^* \wedge f_{\bar{j}_1}^* \wedge \ldots \wedge f_{\bar{j}_s}^*$ respectively.

Notice that we use subscripts for coordinates of forms and superscripts for coordinates of vectors.

Examples.

1. Suppose X and ϕ are as above. Then

$$\bar{\phi} = \sum_{I,J} \overline{\phi_{I\bar{J}}}\ \overline{f_I^*}\ \overline{f_{\bar{J}}^*} = \sum_{I,J} \overline{\phi_{I\bar{J}}}\ f_{\bar{I}}^* f_J^*$$

$$= (-1)^{rs} \sum_{I,J} \overline{\phi_{I\bar{J}}}\ f_J^* f_{\bar{I}}^* .$$

A similar formula holds for $\bar{X}$.

2. Same assumptions on X and ϕ. We have

$$\langle X,\phi\rangle = \sum_{I,J} X^{I\bar{J}} \phi_{I\bar{J}} .$$

Exercises.

1. Show that ${}_cE^*$ $(=({}_cE)^*)$ is naturally isomorphic to ${}_cE'$ and deduce that ${}_cE' \approx E^* \oplus \bar{E}^*$.

2. Let E be a real vector space, F a complex vector space. Show that there is a natural operation of conjugation $S: {}_cE \otimes_{\mathbb{C}} F \to {}_cE \otimes_{\mathbb{C}} F$ that is the identity if $E = \mathbb{R}$ and conjugation on ${}_cE$ if $F = \mathbb{C}$.

3. Show that the set of complex structures on $\mathbb{R}^{2m}$ is in bijective correspondence with $GL(2m,\mathbb{R})/GL(m,\mathbb{C})$ (Here we take the standard complex structure on $\mathbb{R}^{2m}$ and regard $GL(m,\mathbb{C})$ as a subgroup of $GL(2m,\mathbb{R})$).

§5. Generalities on complex vector bundles.

In this section we collect together a number of definitions and elementary facts about complex and holomorphic vector bundles. The reader should be familiar with §5, Chapter 1.

Complexification. Let $E \xrightarrow{p} M$ be a (smooth) real vector bundle over the differential manifold M. The *complexification* ${}_cE$ of E is defined to be the complex vector bundle $E \otimes_{\mathbb{R}} \underline{\mathbb{C}}$ over M. Notice that if E has transition functions $\phi_{ij}: U_{ij} \to GL(m,\mathbb{R})$ then ${}_cE$ has transition functions ${}_c\phi_{ij}: U_{ij} \to GL(m,\mathbb{C})$, where ${}_c\phi_{ij}(x) = \phi_{ij}(x) \otimes_{\mathbb{R}} 1$, $x \in U_{ij}$. We remark that all the results on complexification described in §3 extend immediately to vector bundles and their sections. In particular, we have a conjugation operator S defined on ${}_cE$ and $C^\infty({}_cE)$ and this operator commutes with tensor product operations and duals in the manner outlined in §3.

Complex structures. Let $E \xrightarrow{p} M$ be a (smooth) real vector bundle over the differential manifold M. A *complex structure* J on E is a vector bundle morphism $J: E \to E$ satisfying $J^2 = -I$. Equivalently, a complex structure on E is a C^∞ section J of the vector bundle L(E,E) over M such that $J(x)^2 = -I|E_x$ for all $x \in M$ (see exercise 6, §5, Chapter 1). If E is a complex vector bundle over M then E has a complex structure defined by scalar multiplication by i in the fibres of E. Conversely, it is not hard to show that if E has a complex structure then E has the structure of a complex vector bundle (the proof uses Exercise 3, §4, together with the fact that the quotient map $GL(2m,\mathbb{R}) \to GL(2m,\mathbb{R})/GL(m,\mathbb{C})$ admits local sections).

Definition 5.5.1. Let M be a differential manifold. A complex structure J on $\mathcal{T}M$ is called an *almost complex structure* on M. We refer to M as an *almost complex manifold*.

Example 1. Let M be a complex manifold with atlas $\{(U_i,\phi_i): i \in I\}$. The transition functions for the tangent bundle $\mathcal{T}M$ of M are given by $\phi_{ij} = D(\phi_i\phi_j^{-1})\phi_j$. Since $\phi_i\phi_j^{-1}$ is biholomorphic, $\phi_{ij}: U_{ij} \to GL(m,\mathbb{C})$ for all $i,j \in I$. Hence $\mathcal{T}M$ has the structure of a complex vector bundle and so M has the structure of an almost complex manifold.

Holomorphic and anti-holomorphic vector bundles. For the remainder of this section we shall suppose that M is a complex manifold.

Definition 5.5.2. An m-dimensional *holomorphic vector bundle* E over M consists of a complex manifold E and holomorphic map $p: E \to M$ together with a family $\phi_i: E|U_i \to U_i \times \mathbb{C}^m$ of biholomorphic trivialisations of E.

Remarks.

1. A holomorphic vector bundle necessarily has the structure of a complex vector bundle.

2. An m-dimensional holomorphic vector bundle may equivalently be described by specifying a family $\phi_{ij}: U_{ij} \to GL(m,\mathbb{C})$ of transition functions such that ϕ_{ij} is holomorphic for all i, j. It is clear from the transition function description of holomorphic vector bundles that if E is a holomorphic vector bundle then so is E* and any finite tensor product of tensor, exterior and symmetric powers of E and E*.

3. We denote the space of holomorphic sections of E by $\Omega(M,E)$ or just $\Omega(E)$, if M is implicit from the context.

Example 2. Let M be a complex manifold. Then $\mathcal{T}M$ has the structure of a holomorphic vector bundle.

Suppose that E is a holomorphic vector bundle with transition functions $\phi_{ij}: U_{ij} \to GL(\mathbb{C}^m)$ $(=GL(m,\mathbb{C}))$. If E has complex structure J we let $\bar{E}$ denote the complex vector bundle over M with complex structure -J. The transition functions for $\bar{E}$ are given by

$$\bar{\phi}_{ij}: U_{ij} \to GL(\bar{\mathbb{C}}^m) \ .$$

where $\bar{\phi}_{ij}(x) = \phi_{ij}(x)$, $x \in U_{ij}$, as real linear maps of $\mathbb{C}^m$. Obviously the $\bar{\phi}_{ij}$ are no longer holomorphic maps. Instead they are *anti-holomorphic*. That is, in a local holomorphic coordinate system on M we have $\partial\bar{\phi}_{ij}/\partial z_k = 0$, $1 \le k \le \dim(M)$.

In the sequel we shall say that a complex vector bundle E over M is *anti-holomorphic* if the transition functions for E are anti-holomorphic or, equivalently, if $\bar{E}$ is a holomorphic vector bundle.

Remark. The dual of an anti-holomorphic vector bundle is anti-holomorphic as are finite tensor, exterior and symmetric products.

§6. Tangent and cotangent bundles of a complex manifold.

In this section we show how the theory of §4 can be applied to the study of the tangent and cotangent bundles of a complex manifold. Our results provide the framework we need for the construction of the global Cauchy-Riemann operators on an arbitrary complex manifold that we carry out in §7.

We start this section by briefly indicating how the theory outlined in §2 for differential manifolds can be "complexified".

Let M be a differential manifold with tangent bundle $\mathcal{T}M$. We call the complex vector bundles $\mathcal{T}_c M$ and $\mathcal{T}_c' M$ the *complex tangent* and *complex cotangent bundles* of M respectively. Sections of the bundle $\Lambda^p_c \mathcal{T}'M$ are called complex differential p-forms on M, $p \geq 0$. Exterior differentiation complexifies to give an operator on complex differential forms which we shall continue to denote by d. We note that this operator on complex differential forms obeys all the properties described in §3, or rather their complexified analogues, and in addition is a real operator:

$$\overline{d\phi} = d\bar{\phi}, \ \phi \in C^\infty(\Lambda^p_c \mathcal{T}'M), \ p \geq 0.$$

The Lie bracket complexifies to give a Lie bracket on $C^\infty(\mathcal{T}_c M)$ defined by

$$[X_1 + iY_1, X_2 + iY_2] = [X_1,X_2] - [Y_1,Y_2] - i([Y_1,X_2] + [X_1,Y_2]),$$

where $X_1, X_2, Y_1, Y_2 \in C^\infty(\mathcal{T}M)$. Similarly the Lie derivative complexifies to give a derivation L_Z of the full tensor algebra of $\mathcal{T}_c M$ for all $Z \in C^\infty(\mathcal{T}_c M)$. We remark that the Lie bracket and derivative that we have constructed are real operators, that is they commute with conjugation. For example, if $X,Y \in C^\infty(\mathcal{T}_c M)$ we have $\overline{[X,Y]} = [\bar{X},\bar{Y}]$.

Suppose now that M is an m-dimensional complex manifold with atlas $\{(U_i,\phi_i): i \in I\}$. We let $\phi_{ij} = D(\phi_i\phi_j^{-1})\phi_j: U_{ij} \to GL(m,\mathbb{C})$ denote the transition functions for the tangent bundle $\mathcal{T}M$ of M and J denote the complex structure on $\mathcal{T}M$. Applying the theory of §4, we see that the transition function ${}_c\phi_{ij}$ of $\mathcal{T}_c M$ splits as a sum $\theta_{ij} \oplus \bar{\theta}_{ij}$, where

$\theta_{ij}: U_{ij} \to GL(\mathbb{C}^m)$ and $\bar{\theta}_{ij}: U_{ij} \to GL(\bar{\mathbb{C}}^m)$ (here we have identified ${}_c\mathbb{C}^m$ with $\mathbb{C}^m \oplus \bar{\mathbb{C}}^m$ using the maps j, $\bar{j}$ described in §4). Clearly θ_{ij} and $\bar{\theta}_{ij}$ are the transition functions for complex vector bundles on M which we shall denote by TM and $\overline{TM}$ respectively. By our construction we see that TM and $\overline{TM}$ are complementary complex subbundles of ${}_c\mathcal{T}M$ and so we have ${}_c\mathcal{T}M = TM \oplus \overline{TM}$. Moreover $J = +i$ on TM, $J = -i$ on $\overline{TM}$ and $S(TM) = \overline{TM}$ (Hence the "bar" notation).

We have the natural inclusion map $j: \mathbb{C}^m \to {}_c\mathbb{C}^m \approx \mathbb{C}^m \oplus \bar{\mathbb{C}}^m$ and projection $P: {}_c\mathbb{C}^m \to \mathbb{C}^m$. Since $\theta_{ij} = j\phi_{ij}P$ and j and P are complex linear the θ_{ij} are holomorphic and so TM has the structure of a holomorphic vector bundle. Indeed the map j induces a holomorphic vector bundle isomorphism between $\mathcal{T}M$ and TM. Similarly $\overline{TM}$ has the structure of an anti-holomorphic vector bundle. We call TM the *holomorphic tangent bundle* of M and $\overline{TM}$ the *anti-holomorphic tangent bundle* of M. We reserve these terms for the appropriate *sub*bundles of ${}_c\mathcal{T}M$ and continue to refer to $\mathcal{T}M$ as the real tangent bundle of M even though it is isomorphic to TM.

Taking the standard basis of $\mathbb{C}^m$ we can easily compute the matrices of the transition functions θ_{ij}, $\bar{\theta}_{ij}$. To simplify notation, set $\phi = \phi_i\phi_j^{-1}$, $\theta = \theta_{ij}$ and let $(z_1,\ldots,z_m)$ denote the coordinate system on U_{ij} given by the chart (U_j,ϕ_j). We follow the basis notation given in §4. The qth. column of $[\theta(z)]$ is the vector

$$\begin{aligned}\theta(z)(f_q) &= \theta(z)(\tfrac{1}{2}(e_q \otimes 1 - Je_q \otimes i)\\ &= \tfrac{1}{2}({}_cD\phi_z(e_q \otimes 1) - i\,{}_cD\phi_z(Je_q \otimes 1))\\ &= \tfrac{1}{2}(\partial\phi/\partial x_q - i\partial\phi/\partial y_q) = \partial\phi/\partial z_q\ .\end{aligned}$$

Hence $[\theta] = [\partial\phi_p/\partial z_q]$. Conjugating we have $[\bar{\theta}] = [\partial\bar{\phi}_p/\partial\bar{z}_q] = [\overline{\partial\phi_p/\partial z_q}]$. Notice that our expression for $[\theta]$ given an alternative verification that TM has the structure of a holomorphic vector bundle.

Turning now to dual bundles we let TM* and $\overline{TM}$* denote the complex dual and conjugate complex dual bundles of $\mathcal{T}M$ respectively. We have the direct sum decomposition ${}_c\mathcal{T}^*M = TM^* \oplus \overline{TM}^*$ and, as above, TM* is a holomorphic vector bundle, $\overline{TM}$* an anti-holomorphic vector bundle. We call TM^* and $\overline{TM}^*$ the *holomorphic cotangent* and *anti-holomorphic cotangent* bundles of M respectively. The transition functions ψ_{ij} for ${}_c\mathcal{T}'M$ are given by $\psi_{ij} = {}_c\phi'_{ji}$ and so the transition functions for TM^* and $\overline{TM}^*$ are given by θ^*_{ji} and $\bar{\theta}^*_{ji}$ respectively. In local coordinates, the matrices of θ^*_{ji} and $\bar{\theta}^*_{ji}$ are the transpose and conjugate transpose of the matrix $[\theta_{ij}]$ respectively.

Next we consider the exterior algebras of ${}_c\mathcal{T}M$ and ${}_c\mathcal{T}'M$. Working with transition functions we may construct for $r,s \geq 0$, $r+s = p$, subbundles $\Lambda^{r,s}(M)$ and $\Lambda^{r,s}(M)'$ of $\Lambda^p{}_c\mathcal{T}M$ and $\Lambda^p{}_c\mathcal{T}'M$ respectively such that

$$\Lambda^p{}_c\mathcal{T}M = \bigoplus_{r+s=p} \Lambda^{r,s}(M); \quad \Lambda^p{}_c\mathcal{T}'M = \bigoplus_{r+s=p} \Lambda^{r,s}(M)'.$$

We call $\Lambda^{r,s}(M)$ (resp. $\Lambda^{r,s}(M)'$) the bundle of (r,s)-vectors (resp. (r,s)-forms) on M. If $s = 0$, we see that $\Lambda^{r,0}(M) \approx \Lambda^r TM$ and $\Lambda^{r,0}(M)' \approx \Lambda^r TM^*$. In particular, these bundles are holomorphic vector bundles on M, $r \geq 0$.

Notation. For $r,s,p \geq 0$, we let $C^p(M)$ (resp. $C_p(M)$) denote the space of C^∞ sections of $\Lambda^p{}_c\mathcal{T}'M$ (resp. $\Lambda^p{}_c\mathcal{T}M$). (From now on we shall never need to refer to C^p functions on M unless $p = \infty$, when we write $C^\infty(M)$). We let $C^{r,s}(M)$ (resp. $C_{r,s}(M)$) denote the space of C^∞ sections of $\Lambda^{r,s}(M)'$ (resp. $\Lambda^{r,s}(M)$). We let $\Omega^p(M)$ (resp. $\Omega_p(M)$) denote the space of holomorphic sections of $\Lambda^{p,0}(M)'$ (resp. $\Lambda^{p,0}(M)$).

All the theory described in §4 extends immediately to the exterior algebras of ${}_c\mathcal{T}M$ and ${}_c\mathcal{T}'M$ and the corresponding spaces of sections. In particular for $r,s \geq 0$ we have a conjugation operator $S: \Lambda^{r,s}(M)' \to \Lambda^{s,r}(M)'$ and induced conjugation operator $S: C^{r,s}(M) \to C^{s,r}(M)$ (similarly for (r,s)-vectors). We usually write $S(\phi) = \bar{\phi}$, for ϕ a complex form or vector. Notice that a necessary condition for an (r,s)-form ϕ to be real - $\phi = \bar{\phi}$ - is that $r = s$.

Suppose that N is a complex manifold and $f: M \to N$ is a holomorphic map with tangent map $\mathcal{T}f: \mathcal{T}M \to \mathcal{T}N$. The complexification ${}_c\mathcal{T}f$ of $\mathcal{T}f$ splits as a sum $Tf \oplus \overline{T}f: TM \oplus \overline{TM} \to TN \oplus \overline{TN}$. Similarly the complexification ${}_c\mathcal{T}'f$ of the cotangent map $\mathcal{T}'f: \mathcal{T}'N \to \mathcal{T}'M$ splits as a sum $Tf^* \oplus \overline{Tf}^*: TN^* \oplus \overline{TN}^* \to TM^* \oplus \overline{TM}^*$. Consequently, for $r,s \geq 0$, the tangent and cotangent maps of f induce vector bundle maps

$$\Lambda^{r,s}(f): \Lambda^{r,s}(M) \to \Lambda^{r,s}(N) \text{ and } \Lambda^{r,s}(f)': \Lambda^{r,s}(N)' \to \Lambda^{r,s}(M)'.$$

We now give a local description of complex (r,s)-vectors and forms in terms of the self-conjugate bases given in §4.

Suppose $\phi \in C^{r,s}(M)$ and (V,ζ) is a chart on M. Then $\zeta_*\phi$ is an (r,s)-form on the open subset $\zeta(V)$ of $\mathbb{C}^m$. Set $U = \zeta(V)$ and $\eta = \zeta_*(\phi)$. As is conventional, we denote the standard basis of $\mathbb{C}^{m'}$ by $\{dx_1, dy_1, \ldots, dx_m, dy_m\}$. For $1 \le j \le m$ we set $dz_j = dx_j + idy_j \in \mathbb{C}^{m*}$, $d\bar{z}_j = dx_j - idy_j \in \bar{\mathbb{C}}^{m*}$. Then $\{dz_j, d\bar{z}_j: 1 \le j \le m\}$ is the self-conjugate basis of $\mathbb{C}^{m*} \oplus \bar{\mathbb{C}}^{m*}$ described in §4. Moreover, if we think of dz_j, $d\bar{z}_j$ as defining sections of $T\mathbb{C}^{m*}$, $\overline{T\mathbb{C}}^{m*}$ respectively, $\{dz_j: 1 \le j \le m\}$ and $\{d\bar{z}_j: 1 \le j \le m\}$ give bases for $C^\infty(TU^*)$ and $C^\infty(\overline{TU}^*)$ over $C^\infty(U)$ respectively. Hence we may write $\eta \in C^{r,s}(U)$ uniquely in the form

$$\eta = \sum_{I,J} \eta_{I\bar{J}}\, dz_I dz_{\bar{J}} \,,$$

where $\eta_{I\bar{J}} \in C^\infty(U)$ and the summation over the r-tuples I and s-tuples J is as described in §4. Next we turn to the local form for complex (r,s)-vectors. Identifying (complex) vector fields with (complex) derivations, it is clear that if we set $\partial/\partial z_j = \frac{1}{2}(\partial/\partial x_j - i\partial/\partial y_j)$ and $\partial/\partial \bar{z}_j = \frac{1}{2}(\partial/\partial x_j + i\partial/\partial y_j)$ then $\{\partial/\partial z_j: i \le j \le m\}$ and $\{\partial/\partial \bar{z}_j: i \le j \le m\}$ form bases over $C^\infty(U)$ for $C^\infty(TU)$ and $C^\infty(\overline{TU})$ respectively. Hence we may write $X \in C_{r,s}(U)$ uniquely in the form

$$X = \sum_{I,J} X^{I\bar{J}} \partial/\partial z_I \; \partial/\partial z_{\bar{J}} \,,$$

where $X^{I\bar{J}} \in C^\infty(U)$ and we again follow the notational conventions of §4.

Remark. Much of what we have done in this section goes over to almost complex manifolds. Thus if M is an almost complex manifold with almost complex structure J we may define $TM = \text{Kernel}(J-i)$ and $\overline{TM} = \text{Kernel}(J+i)$. TM and $\overline{TM}$ are complementary complex subbundles of $\mathcal{T}_c M$ though now, of course, we can no longer say that TM is a holomorphic vector bundle as we are not assuming that M has a complex structure. In the next section we shall discuss the important question of when an almost complex structure on M is associated to a complex structure on M.

§7. Calculus on a complex manifold.

Suppose that M is an almost complex manifold with complex structure J on M. We start this section by investigating the relationships, if any, between J and exterior differentiation and Lie brackets.

Definition 5.7.1. The *torsion* of the almost complex structure J on M is the tensor field $N \in C^{\infty}(\Lambda^2 \mathcal{T}^* M \otimes \mathcal{T}M)$ characterised by

$$\langle N, X \wedge Y \rangle = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y],$$

where $X,Y \in C^{\infty}(\mathcal{T}M)$.

Remarks.

1. As usual the reader should verify, using exercise 6, §5, Chapter 1, that N is a well-defined tensor field on M.

2. In the literature N is usually defined as a section of $\otimes^2 \mathcal{T}^* M \otimes \mathcal{T}M$ and differs from the torsion field as we have defined it by a factor of 4.

3. In the sequel we usually abbreviate an expression like $\langle N, X \wedge Y \rangle$ to $N(X \wedge Y)$ or just $N(X,Y)$. Note that the pairing is that between exterior and not tensor powers.

The significance of the torsion of an almost complex structure may be gauged from

Proposition 5.7.2. The spaces $C^{\infty}(TM)$, $C^{\infty}(\overline{TM})$ are Lie subalgebras of $C^{\infty}(\mathcal{T}_c M)$ if and only if the torsion tensor field N vanishes.

Proof. Let $U,V \in C^{\infty}(TM)$. We may write U, V uniquely in the form $U = X - iJX$, $V = Y - iJY$, $X,Y \in C^{\infty}(\mathcal{T}M)$. Computing we see that $[U,V] = A + iB$, where $A = [X,Y] - [JX,JY]$, $B = [JX,Y] + [X,JY]$. Now $A + iB \in C^{\infty}(TM)$ iff $JB = A$. That is, iff $J[JX,Y] + J[X,JY] = [X,Y] - [JX,JY]$ But this is precisely the condition that the torsion field N vanishes. Conjugating we see that if one of $C^{\infty}(TM)$, $C^{\infty}(\overline{TM})$ is a Lie subalgebra of $C^{\infty}(\mathcal{T}_c M)$ so is the other. □

Proposition 5.7.3. The torsion of the almost complex structure assoicated to a complex manifold vanishes.

Proof. Choose local analytic coordinates and compute N on the basis vector fields $\partial/\partial x_j$, $\partial/\partial y_k$, $1 \leq j,k \leq m$. The Lie bracket of any pair of these constant fields vanishes and since $J(\partial/\partial x_j) = \partial/\partial y_j$, we see that N must vanish identically. □

Remark. It is true, by a fundamental theorem of Newlander and Nirenberg, that if the torsion of an almost complex structure on M vanishes then the almost complex structure is associated to a complex structure on M. We say that the almost complex structure is *integrable*. This result is not difficult to prove if M is real analytic (a proof may be found in Kobayashi and Nomizu [2; Appendix 8]). For the general case we refer to Hörmander [1]. Although we shall not make any systematic study of almost complex manifolds in these notes we should point out that there are topological obstructions on a differential manifold for it to admit an almost complex structure and on an almost complex manifold for it to admit an integrable complex structure. Specifically, a theorem of Hirzebruch and Hopf [1] gives necessary and sufficient conditions on a compact, oriented 4-manifold for it to admit an almost complex structure. These conditions imply, for example, that S^4 does not admit an almost complex structure and so cannot be given the structure of a complex manifold. Borel and Serre [1] prove that S^n can admit an almost complex structure only if $n = 2,4,6$. Of course, if $n = 2$ we obtain the Riemann sphere. This leaves the case $n = 6$. It is well known that S^6 admits an almost complex structure (see Kobayashi and Nomizu [2; page 139]) which is, however, not integrable. As yet it is unknown whether S^6 admits an integrable almost complex structure. Results of Van der Ven [1] show that there are topological obstructions to the existence of integrable almost complex structures on an almost complex manifold. For a useful survey of results on 4-manifolds see Pittie [1].

Theorem 5.7.4. Let M be a complex manifold. Then for $r,s \geq 0$ we have

$$d(C^{r,s}(M)) \subset C^{r+1,s}(M) + C^{r,s+1}(M).$$

Proof. Let $\phi \in C^p(M)$ and $X_0,\ldots,X_p \in C^\infty(\mathcal{T}M)$. Then, as in §2, we have

$$\langle d\phi, X_0 \wedge \ldots \wedge X_p\rangle = \sum_{i=0}^{p} (-1)^i L_{X_i} \langle \phi, X_0 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_p\rangle$$

$$+ \sum_{0 \le i < j} (-1)^{i+j} \langle \phi, [X_i, X_j] \wedge X_0 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge \hat{X}_p\rangle .$$

Now suppose $p = r+s$, ϕ is an (r,s)-form and each X_j lies in $C^\infty(TM)$ or $C^\infty(\overline{TM})$. By Proposition 5.7.2 we see that if more than $(r+1)$ of the X_j's lie in $C^\infty(TM)$ or more than $(s+1)$ of the X_j's lie in $C^\infty(\overline{TM})$ then the right hand side of the above expression vanishes (remember that the pairings between TM, $\overline{TM}^*$ and $\overline{TM}$, TM^* are zero). □

Remark. If M is an almost complex manifold it is easily seen that $d(C^{r,s}(M)) \subset C^{p+2,q-1}(M) + C^{p+1,q}(M) + C^{p,q+1}(M) + C^{p-1,q+2}(M)$. A straightforward calculaion shows that the result of Theorem 5.7.4 holds if and only if the torsion of the almost complex structure vanishes. See Kobayashi and Nomizu [2] for further details.

It follows from Theorem 5.7.4 that if M is a complex manifold we can define for $r,s \ge 0$ operators $\partial: C^{r,s}(M) \to C^{r+1,s}(M)$ and $\bar\partial: C^{r,s}(M) \to C^{r,s+1}(M)$ characterised by the identity $d = \partial + \bar\partial$. We remark also that for $p \ge 0$, ∂, $\bar\partial$ induce operators $\partial, \bar\partial: C^p(M) \to C^{p+1}(M)$ satisfying $d = \partial + \bar\partial$.

Properties of the operators ∂, $\bar\partial$.

1. $\partial^2 = 0$, $\bar\partial^2 = 0$, $\partial\bar\partial + \bar\partial\partial = 0$.

2. ∂, $\bar\partial$ are conjugate operators: $\bar\partial\phi = \overline{\partial\bar\phi}$, $\phi \in C^p(M)$.

3. $\partial(\phi \wedge \zeta) = \partial\phi \wedge \zeta + (-1)^p \phi \wedge \partial\zeta$, $\phi \in C^p(M)$, $\zeta \in C^q(M)$. Similarly for $\bar\partial$.

4. If $f: M \to N$ is holomorphic and $\phi \in C^p(N)$ then $f^*\partial\phi = \partial(f^*\phi)$. Similarly for $\bar\partial$.

5. In local coordinates, suppose $\phi = \sum_{I,J} \phi_{I\bar J} dz_I dz_{\bar J}$. Then

$$\partial\phi = \sum_{j=1}^{m} \sum_{I,J} \partial\phi_{I\bar J}/\partial z_j \, dz_j \, dz_I dz_{\bar J} \text{ and } \bar\partial\phi = \sum_{j=1}^{m} \sum_{I,J} \partial\phi_{I\bar J}/\partial \bar z_j \, dz_j dz_I dz_{\bar J}.$$

6. For $p \geq 0$, we have $\Omega^p(M) = \text{Kernel } \bar{\partial}: C^{p,0}(M) \to C^{p,1}(M)$. In particular, $A(M) = \text{Kernel } \bar{\partial}: C^\infty(M) \to C^{0,1}(M)$.

Properties 1-4 follow immediately from the corresponding properties of d together with Theorem 5.7.4. For property 5 we use the local identity $df = \partial f/\partial z_j \, dz_j + \partial f/\partial \bar{z}_j d\bar{z}_j$, f a C^∞ function. Property 6 is immediate from Property 5.

We see from Properties 5 and 6 that the operator $\bar{\partial}$ is our required generalisation and globalization of the Cauchy-Riemann equationsdiscussed in Chapter 2. We observe that for $p \geq 0$ we have the sequences

$$\Omega^p(M) \hookrightarrow C^{p,0}(M) \xrightarrow{\bar{\partial}} C^{p,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} C^{p,m}(M) \to 0 .$$

Since $\bar{\partial}^2 = 0$, we may for $p,q \geq 0$ define the vector spaces

$$H^{p,q}_{\bar{\partial}}(M) = (\text{Kernel } \bar{\partial}: C^{p,q}(M) \to C^{p,q+1}(M))/\bar{\partial}C^{p,q-1}(M).$$

These spaces measure the degree of unsolvability of our generalised Cauchy-Riemann equations. Like the de Rham groups they turn out to be important invariants though now they reflect analytic rather than topological properties of a complex manifold. We return to these matters in the next Chapter.

Example 1. Let $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$. Then $\alpha \wedge \beta \in \Omega^{p+q}(M)$. Indeed by properties 3 and 6 we have $\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^p \alpha \wedge \bar{\partial}\beta = 0$. Hence $\alpha \wedge \beta \in \Omega^{p+q}(M)$.

We conclude this section by extending $\bar{\partial}$ to holomorphic vector bundle valued differential forms. Suppose that E is a holomorphic vector bundle on M. For $r,s \geq 0$, we let $\Lambda^{r,s}(M,E)'$ denote the complex vector bundle $\Lambda^{r,s}(M)' \otimes E$ and $C^{r,s}(M,E)$ denote the space of C^∞ sections of $\Lambda^{r,s}(M)' \otimes E$. We claim that for $r,s \geq 0, \bar{\partial}$ extends to a map

$$\bar{\partial}_E: C^{r,s}(M,E) \to C^{r,s+1}(M,E).$$

To construct $\bar{\partial}_E$ we work locally. Suppose that $\theta_U: E|U \to U \times \mathbb{C}^p$ is a trivialisation of E over the open subset U of M and that we are given a complex analytic coordinate system on U. Given $s \in C^{r,s}(M,E)$, set $s_U = s|U$. We have

$$s_U^a = \sum_{I,J} s^a_{U,I\bar{J}}\, dz_I d\bar{z}_{\bar{J}},\ a = 1,\dots,p,$$

where $s_U = (s_U^1,\dots,s_U^p)\colon U \to \Lambda^{r,s}(\mathbb{C}^m)' \otimes \mathbb{C}^p$. We define $\bar{\partial}_E s$ by

$$(\bar{\partial}_E s)_U^a = \sum_{j=1}^{m} \sum_{I,J} \partial s^a_{U,I\bar{J}}/\partial \bar{z}_j\, d\bar{z}_j dz_I d\bar{z}_{\bar{J}},\ a = 1,\dots,p.$$

We must check that our definition of $\bar{\partial}_E s$ does not depend on our choices of trivialisation of E. Suppose that $\theta_V\colon E|V \to V \times \mathbb{C}^p$ is another trivialisation of E and let θ_{UV} denote the transition function associated to the trivialisations θ_U and θ_V. On $U \cap V$ we have

$$s_U = \theta_{UV} s_V.$$

Since θ_{UV} is analytic we see that $\partial\theta_{UV}/\partial\bar{z}_j = 0$, $j = 1,\dots,m$, and so $(\bar{\partial}_E s)_U = \theta_{UV}(\bar{\partial}_E s)_V$. Hence $\bar{\partial}_E s$ is a well defined section of $\Lambda^{r,s+1}(M,E)'$.

If E is an anti-holomorphic vector bundle on M we may similarly define an operator $\partial_E\colon C^{r,s}(M,E) \to C^{r+1,s}(M,E)$, $r,s \geq 0$. Indeed, using conjugation we may define $\partial_E\phi = \overline{(\bar{\partial}_E\bar{\phi})}$, $\phi \in C^{r,s}(M,E)$, where conjugation is induced from the conjugation map $S\colon \Lambda^{r,s}(M,E)' \to \Lambda^{s,r}(M,\bar{E})'$ (see Exercise 2, §4).

In the sequel we usually drop the subscripts from ∂_E and $\bar{\partial}_E$ and just write ∂ and $\bar{\partial}$.

Properties of the operators ∂, $\bar{\partial}$ on bundle valued forms. We shall only state properties for $\bar{\partial}$; those for ∂ follow by conjugation. In what follows we assume that E is a holomorphic vector bundle on M.

1. $\bar{\partial}^2 = 0$.

2. Kernel $\bar{\partial}\colon C^{p,0}(M,E) \to C^{p,1}(M,E)$ is the space $\Omega^p(M,E)$ of holomorphic sections of $\Lambda^p TM^* \otimes E$. In particular, Kernel $\bar{\partial}\colon C^\infty(E) \to C^{0,1}(M,E)$ is the space $\Omega(M,E)$ of holomorphic sections of E.

3. $\bar{\partial}$ commutes with contractions on finite tensor products of tensor, exterior and symmetric powers of E and E*. For example, if

$\phi \in C^{r,s}(M,\Lambda^p E \otimes \Lambda^q E^*)$ and $p > q$, $\mathbb{C}(\bar\partial\phi) = \bar\partial(\mathbb{C}\phi) \in C^{r,s}(M,\Lambda^{p-q}E)$. Here $\mathbb{C}$ is induced from the contraction $\mathbb{C}: \Lambda^p E \otimes \Lambda^q E^* \to \Lambda^{p-q}E$.

4. $\bar\partial$ commutes with contractions between the bundles $\Lambda^{r,s}(M)'$ and exterior powers of the holomorphic tangent bundle of M. That is, if $\phi \in C^{r,s}(M,\Lambda^p TM)$ and $r \geq p$, we have $\bar\partial(\mathbb{C}\phi) = \mathbb{C}(\bar\partial\phi) \in C^{r-p,s}(M)$.

These properties all follow immediately from our local description of $\bar\partial$.

As a consequence of property 1 we have for $p \geq 0$ the sequences

$$\Omega^p(M,E) \hookrightarrow C^{p,0}(M,E) \xrightarrow{\bar\partial} C^{p,1}(M,E) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} C^{p,m}(M,E) \to 0$$

and the corresponding vector spaces

$$H^{p,q}_{\bar\partial}(M,E) = (\text{Kernel } \bar\partial: C^{p,q}(M,E) \to C^{p,q+1}(M,E))/\bar\partial C^{p,q-1}(M,E).$$

Examples.

2. Let E be a holomorphic vector bundle and $J \in C^\infty(L(E,E))$ denote the complex structure on E. Then $\bar\partial J = 0$ and so J is a holomorphic section of L(E,E). That $\bar\partial J = 0$ is a consequence of the fact that locally J is a constant section. For the same reason $\bar\partial I = 0$, where I is the identity section.

3. Let $X \in C^\infty(TM)$, $Y \in C^\infty(\overline{TM})$. Then $[X,Y] = \mathbb{C}_X\partial Y - \mathbb{C}_Y\bar\partial X$. We shall give an invariant proof of this identity using property 4 above. Let $f \in C^\infty(M)$, then

$$\begin{aligned} L_{[X,Y]}f &= \langle d\langle df,Y\rangle,X\rangle - \langle d\langle df,X\rangle,Y\rangle \\ &= \langle \partial\langle \bar\partial f,Y\rangle,X\rangle - \langle \bar\partial\langle \partial f,X\rangle,Y\rangle \\ &= (\mathbb{C}_X\mathbb{C}_Y\partial\bar\partial f - \mathbb{C}_Y\mathbb{C}_X\bar\partial\partial f) + (\langle \bar\partial f,\mathbb{C}_X\partial Y\rangle - \langle \partial f,\mathbb{C}_Y\bar\partial X\rangle), \\ &= \langle df,\mathbb{C}_X\partial Y - \mathbb{C}_Y\bar\partial X\rangle, \text{ proving our assertion.} \end{aligned}$$

Exercises.

1. Starting with the local description of $\bar\partial$ (property 5), prove directly that if f is a holomorphic map then $\bar\partial(f^*\phi) = f^*(\bar\partial\phi)$ (property 4).

Deduce that $\bar\partial$ may be defined invariantly on complex differential forms on a complex manifold.

2. Let E be a holomorphic vector bundle on M. Suppose $s \in C^\infty(E)$, $f \in C^\infty(M)$. Prove $\bar\partial(fs) = \bar\partial f \otimes s + f\bar\partial s$. More generally, if $\phi \in C^{r,s}(M)$, prove that $\bar\partial(\phi \otimes s) = \bar\partial\phi \otimes s + (-1)^{r+s}\phi \wedge \bar\partial s$.

§8. The Dolbeault-Grothendieck Lemma.

This section is devoted to the proof of an important result that plays the same rôle in the theory of complex manifolds as the Poincaré lemma does in the cohomology of differential manifolds.

Theorem 5.8.1. (Dolbeault-Grothendieck lemma). Let D be an open polydisc in $\mathbb{C}^n$ and suppose that $f \in C^{p,q+1}(D)$ satisfies $\bar\partial f = 0$ $(p,q \geq 0)$. Then if W is any relatively compact open subset of D there exists $u \in C^{p,q}(W)$ such that $\bar\partial u = f$ on W.

Proof. The theorem is proved inductively. The kth. step of the induction is to prove the theorem true if f is independent of $d\bar z_{k+1},\dots,d\bar z_n$. The theorem is trivially true when $k = 0$ and the theorem is obtained for $k = n$.

Let us assume that the theorem has been proved for $k - 1$ and that f does not involve $d\bar z_{k+1},\dots,d\bar z_n$. We may write f uniquely in the form

$$f = d\bar z_k \wedge g + h,$$

where $g \in C^{p,q}(D)$, $h \in C^{p,q+1}(D)$ and g and h are independent of $d\bar z_k,\dots,d\bar z_n$. Set $g = \sum_{I,J} g_{I\bar J}dz_I d\bar z_{\bar J}$. Since $\bar\partial f = 0$, we have

$$\partial g_{I\bar J}/\partial\bar z_j = 0,\ j > k \qquad \dots.A$$

We now find a solution $G_{I\bar J}$ of the equation

$$\partial G_{I\bar J}/\partial\bar z_k = g_{I\bar J}\ .$$

To do this suppose $D = \prod_{j=1}^{n} D_j$, $D_j \subset \mathbb{C}$, and choose $\phi \in C_c^\infty(D_k)$ such that $\phi(z_k) = 1$ on a neighbourhood W' of $\bar W$. Define

$$G_{I\bar{J}}(z) = (2\pi i)^{-1}\int_{\mathbb{C}} (t-z_k)^{-1}\phi(t)g_{I\bar{J}}(z_1,\ldots,z_{k-1},t,z_{k+1},\ldots,z_n)dtd\bar{t}$$

$$= -(2\pi i)^{-1}\int_{\mathbb{C}} t^{-1}\phi(z_k-t)g_{I\bar{J}}(z_1,\ldots,z_{k-1},t,z_{k+1},\ldots,z_n)dtd\bar{t}.$$

The second expression for $G_{I\bar{J}}$ implies that $G_{I\bar{J}} \in C^{\infty}(D)$. Theorem A1.6 implies that $\partial G_{I\bar{J}}/\partial\bar{z}_k = g_{I\bar{J}}$ on W' and, by (A), that $\partial G_{I\bar{J}}/\partial\bar{z}_j = 0$, $j > k$. Set $G = \sum_{I,J} G_{I\bar{J}}dz_I d\bar{z}_{\bar{J}}$. Then on W' we have $\bar{\partial}G = d\bar{z}_k \wedge g + h_1$, where h_1 is independent of $d\bar{z}_k,\ldots,d\bar{z}_n$. Hence, on W', $h - h_1 = f - \bar{\partial}G$ is independent of $d\bar{z}_k,\ldots,d\bar{z}_n$. Since $\bar{\partial}(h-h_1) = 0$, we may apply the inductive hypothesis to find $v \in C^{p,q}(W)$ such that $\bar{\partial}v = f - \bar{\partial}G$ on W. Setting $u = v + g$, we have $\bar{\partial}u = f$ on W. □

Remark. It is easy to see, using bump functions, that we may construct $u \in C^{p,q}(D)$ satisfying $\bar{\partial}u = f$ on W.

Theorem 5.8.1 is sufficient for the development of cohomology theory in Chapter 6. However, we shall now prove a stronger version of the Dolbeault-Grothendieck Lemma and obtain a particular case of a result that holds on arbitrary Stein manifolds and to which we shall return in Chapter 11.

Theorem 5.8.2. Let D be an open, not necessarily relatively compact, polydisc in $\mathbb{C}^n$ and suppose $f \in C^{p,q+1}(D)$ satisfies $\bar{\partial}f = 0$. Then there exists $u \in C^{p,q}(D)$ such that $\bar{\partial}u = f$. Here we assume $p,q \geq 0$.

Proof. We divide the proof into two cases: $q = 0$, $q \geq 1$.

Case 1. $q = 0$. Choose a sequence D_j, $j \geq 1$, of relatively compact open polydiscs in $\mathbb{C}^n$ which have the same centres as D and which satisfy:

A. $\bar{D}_j \subset D_{j+1}$, $j \geq 1$ and B. $\bigcup_{j\geq 1} D_j = D$.

If $u = \sum u_I dz_I \in C^{p,0}(D)$ and $K \subset D$, we define $|u|_K = \max_I \|u_I\|_K$.

We shall construct inductively a sequence $u_j \in C^{p,0}(D)$ satisfying

1. $\bar{\partial}u_j = f$ on some open neighbourhood of $\bar{D}_j$, $j \geq 1$.

2. $|u_{j+1} - u_j|_{D_j} < 2^{-j}$, $j \geq 1$.

The existence of u_1 follows from Theorem 5.8.1. Assume that we have constructed $u_1,\ldots,u_k$ satisfying the conditions above. By Theorem 5.8.1, there exists $u'_{k+1} \in C^{p,0}(D)$ such that $\bar{\partial}u'_{k+1} = f$ on an open neighbourhood of $\bar{D}_{k+1}$. The difference $u'_{k+1} - u_k$ is holomorphic on an open neighbourhood of $\bar{D}_k$ since $\bar{\partial}(u'_{k+1} - u_k) = f - f = 0$ on an open neighbourhood of $\bar{D}_k$. Hence, if we take Taylor's expansion of the coefficients of $u'_{k+1} - u_k$ at the centre of D, we may find $P \in \Omega^p(D)$, with polynomial coefficients, such that

$$|u'_{k+1} - u_k - P|_{\bar{D}_k} < 2^{-k} .$$

Now we define $u_{k+1} = u'_{k+1} - P$ and see that u_{k+1} satisfies the required conditions and so the inductive step is completed.

The sequence (u_k) has coefficients which converge uniformly on each $\bar{D}_k$ and so (u_k) converges to a continuous (p,0)-form, u. Now $u = u_1 + \sum_{j=1}^{\infty} (u_{j+1} - u_j)$ and only finitely many of the differences $u_{j+1} - u_j$ are not holomorphic on any given D_k. Hence, by Corollary 2.1.8, u must be C^∞ on each D_k and so $u \in C^{p,0}(D)$. Finally, on each D_k, $u = u_k + a_k$, $a_k \in A(D_k)$, and so $\bar{\partial}u = \bar{\partial}u_k = f$ on each D_k. Hence $\bar{\partial}u = f$ on D.

Case 2. $q \geq 1$. We choose a sequence D_j of polydiscs in $\mathbb{C}^n$ satisfying the conditions of Case 1. We shall construct inductively a sequence $u_j \in C^{p,q}(D)$ such that

1. $f = \bar{\partial}u_j$ on some open neighbourhood of $\bar{D}_j$, $j \geq 1$.

2. $u_{j+1}|D_j = u_j$, $j \geq 1$.

The existence of u_1 follows from Theorem 5.8.1. Suppose we have constructed $u_1,\ldots,u_k$ satisfying the conditions above. By Theorem 5.8.1, there exists $u'_{k+1} \in C^{p,q}(D)$ such that $\bar{\partial}u'_{k+1} = f$ on some open neighbourhood of $\bar{D}_{k+1}$. Now $\bar{\partial}(u'_{k+1} - u_k) = 0$ on some open neighbourhood of $\bar{D}_k$ and so, since $q \geq 1$, another application of Theorem 5.8.1 implies that there exists $\phi \in C^{p,q-1}(D)$ such that $u'_{k+1} - u_k = \bar{\partial}\phi$ on some open neighbourhood W of D_k. Define $u_{k+1} = u'_{k+1} - \bar{\partial}\phi$. Then, since $\bar{\partial}^2 = 0$, we see that $\bar{\partial}u_{k+1} = \bar{\partial}u'_{k+1} = f$ on an open neighbourhood of $\bar{D}_{k+1}$ and $u_{k+1}|\bar{D}_k = u'_{k+1} - \bar{\partial}\phi = u_k$. This completes the inductive step and we now define $u \in C^{p,q}(D)$ by $u|D_j = u_j$, $j \geq 1$. Clearly $\bar{\partial}u = f$. □

Remarks. Both Theorems 5.8.1 and 5.8.2 hold for $\mathbb{C}^m$-valued forms. The proofs are identical to those given above.

Corollary 5.8.3. Every open polydisc in $\mathbb{C}^n$ is a Cousin I, II, A and B domain. In particular, $\mathbb{C}^n$ is a Cousin I, II, A and B domain.

Proof. Propositions 2.7.1 and 2.7.3. □

Corollary 5.8.4. Every holomorphic line bundle on an open polydisc in $\mathbb{C}^n$ is holomorphically trivial.

Proof. Let $\phi_{ij}: U_{ij} \to \mathbb{C}^{\theta}$ be the transition functions for the holomorphic line bundle L on the polydisc D. Then $\phi_{ij}\cdot\phi_{jk} = \phi_{ik}$ for all i, j, k and so $\{\phi_{ij}\}$ is the data for a Cousin B problem on D. By Corollary 5.8.3, there exists $a_i \in A^*(U_i)$ such that $\phi_{ij} = a_j/a_i$. But, by §5 of Chapter 1, this implies that L is holomorphically trivial. □

Exercises.

1. Suppose $p,q \geq 0$ and $m > 1$. Show that if $f \in C_c^{p,q+1}(\mathbb{C}^m)$ and $\bar{\partial}f = 0$ then there exists $u \in C_c^{p,q}(\mathbb{C}^m)$ such that $\bar{\partial}u = f$.

2. Show that the open Euclidean disc $E(z;r)$ in $\mathbb{C}^n$ is a Cousin I, II, A and B domain (Use Exercise 1, §1, Chapter 2).

§9. Holomorphic vector bundles on compact complex manifolds.

In this section we present a number of important examples of holomorphic vector bundles on compact complex manifolds. We shall pay particular attention to the spaces of holomorphic sections of such bundles which, by the theory of §7, may be represented as the kernel of the $\bar{\partial}$-operator.

We start by proving an elementary special case of a rather general finiteness theorem that we return to in Chapter 7.

Theorem 5.9.1. Let E be a holomorphic vector bundle on the compact complex manifold M. Then $\dim_{\mathbb{C}}\Omega(E) < \infty$.

Proof. Suppose that we are given a finite open cover $\{U_i: i=1,\ldots,r\}$ of M such that over each U_i, E has a holomorphic trivialisation $\theta_i: E|U_i \to U_i \times \mathbb{C}^p$, $p = \dim(E)$. Suppose also that $\{V_i\}$

is an open refinement of the cover $\{U_i\}$ such that for all i, V_i is a relatively compact subset of U_i. Given $s \in \Omega(E)$, we let $s_i: V_i \to \mathbb{C}^p$ denote the local representative of s on V_i.

If $\| \ \|$ denotes the standard Euclidean norm on $\mathbb{C}^p$, we may define a norm $| \ |$ on $\Omega(E)$ by

$$|s| = \sum_{i=1}^{r} \|s_i\|_{V_i}, \quad s \in \Omega(E).$$

Observe that $\|s_i\|_{V_i} < \infty$ since $\bar{V}_i$ is compact. We shall prove that the closed unit ball B in the normed vector space $(\Omega(E), | \ |)$ is compact. This implies that $(\Omega(E), | \ |)$ is locally compact and hence finite dimensional by F. Riesz' theorem (for an elementary proof of Riesz' theorem see Field [1; page 54]). Suppose then that (s^j) is a sequence in B. Given i, $1 \le i \le r$, we have corresponding sequences $(s_i^j) \subset A(V_i, \mathbb{C}^p)$ of local representatives. By our definition of $| \ |$, it is clear that for all i we have

$$\|s_i^j(z)\| \le 1, \; z \in V_i, \; j \ge 1.$$

In particular the sequence (s_1^j) is bounded on $\bar{V}_1$ and so by Montel's theorem (Theorem 2.1.9) we may find a subsequence $(t(1)^j)$ of (s^j) which converges uniformly on V_1. Proceeding inducitvely, suppose that we have constructed a subsequence $(t(k)^j)$ of (s^j) which converges uniformly on $V_1 \cup \ldots \cup V_k$. Applying Montel's theorem we may find a subsequence $(t(k+1)^j)$ of $(t(k)^j)$ which converges uniformly on V_{k+1} and hence uniformly on $V_1 \cup \cdots \cup V_{k+1}$. Hence we may find a subsequence (t^j) of (s^j) which converges uniformly on $V_1 \cup \ldots \cup V_r = M$. Hence B is sequentially compact and therefore compact. □

Holomorphic vector bundles on projective space.

To each point $\ell \in P^n(\mathbb{C})$ is naturally associated a complex line in $\mathbb{C}^{n+1}$. This suggests that we should be able to construct a complex line bundle over $P^n(\mathbb{C})$ whose fibre at the point $\ell \in P^n(\mathbb{C})$ is the line $\ell \subset \mathbb{C}^{n+1}$. We start this subsection by constructing this "tautological" line bundle (see also Exercise 2, §3, Chapter 4).

Let $L \subset P^n(\mathbb{C}) \times \mathbb{C}^{n+1}$ denote the set $\{(\ell,z): z \in \ell\}$. The projection on $P^n(\mathbb{C})$ induces a projection $\pi: L \to P^n(\mathbb{C})$. Recalling §7 of Chapter 4, we see that L is $\mathbb{C}^{n+1}$ blown up at the origin and so, in particular, L has the structure of a complex manifold of dimension n + 1 and π is holomorphic. We claim that L has the natural structure of a holomorphic line bundle over $P^n(\mathbb{C})$. For this we have only to observe that the maps $\theta_i: L|U_i \to U_i \times \mathbb{C}$ defined by

$$\theta_i((z_0,\ldots,z_n),(a_0,\ldots,a_n)) = ((z_0,\ldots,z_n),a_i)$$

define holomorphic trivialisations for L. The corresponding transition functions $\theta_{ij}: U_{ij} \to GL(1,\mathbb{C}) = \mathbb{C}^{\bullet}$ are given by

$$\theta_{ij}(z_0,\ldots,z_n) = z_i/z_j.$$

We call the holomorphic line bundle L the *tautological* or *universal* line bundle on $P^n(\mathbb{C})$.

We let H denote the dual bundle L^* of L. For reasons that will soon become clear we call H the *hyperplane section bundle* of $P^n(\mathbb{C})$. For $p \in \mathbb{Z}$ we define

$$H^p = \otimes^p H, \; p \geq 0$$
$$= \otimes^{-p} L, \; p \leq 0 .$$

Note that the transition functions θ^p_{ij} for H^p are given by $\theta^p_{ij} = (z_j/z_i)^p$.

The next proposition gives an indication of the important rôle that the hyperplane section bundle plays in projective algebraic geometry and also indicates an important bridge that exists between complex analysis and algebra.

Proposition 5.9.2. For $p \geq 0$, $\Omega(H^p)$ is canonically isomorphic to the space $P^{(p)}(\mathbb{C}^{n+1})$ of homogeneous polynomials of degree p on $\mathbb{C}^{n+1}$. For $p < 0$, $\Omega(H^p)$ consists of the zero section.

Proof. Suppose $p \geq 0$. If $s \in \Omega(H^p)$, we let $s_i: U_i \to \mathbb{C}$ denote the local representatives of s relative to the standard trivialisation of H^p. For $0 \leq i, j \leq n$ we have $\theta^p_{ij}s_j = s_i$ and so

$$s_j(z_0,\ldots,z_n)z^p_j = s_i(z_0,\ldots,z_n)z^p_i .$$

Hence we may define the holomorphic map $S: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ by $S(z_0,\dots,z_n) = s_i(z_0,\dots,z_n)z_i^p$, $z_i \neq 0$. Since the s_i are homogeneous of degree zero, S is homogeneous of degree p. By Hartog's theorem, S extends to $\mathbb{C}^{n+1}$ as an analytic function which we continue to denote by S. Since S is homogeneous of degree p, we see by taking Taylor's series at the origin of $\mathbb{C}^{n+1}$, that S must be a homogeneous polynomial of degree p. Hence we have defined a map of $\Omega(H^p)$ into $P^{(p)}(\mathbb{C}^{n+1})$, $p \geq 0$. If $S \in P^{(p)}(\mathbb{C}^{n+1})$, define $s_i = S/z_i^p$, $i = 0,\dots,n$. The s_i are the local representatives of a holomorphic section of H^p. Since the maps between $\Omega(H^p)$ and $P^{(p)}(\mathbb{C}^{n+1})$ are clearly inverses of one another we have shown that $\Omega(H^p)$ is canonically isomorphic to $P^{(p)}(\mathbb{C}^{n+1})$.

We leave the case $p < 0$ as an exercise for the reader which makes use of the isomorphism $H^p \otimes H^{-p} \approx \underline{\mathbb{C}}$. □

As a special case of the proposition we see that if $p = 1$, sections of H correspond to linear functionals on $\mathbb{C}^{n+1}$. In particular the zero sets of such sections are hyperplanes in $P^n(\mathbb{C})$.

Next we show that there exist natural exact sequences

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow (n+1)H \longrightarrow TP^n \longrightarrow 0$$

$$0 \longrightarrow TP^{n*} \longrightarrow (n+1)L \longrightarrow \underline{\mathbb{C}} \longrightarrow 0 .$$

Here (n+1)H denotes the (n+1)-fold direct sum of H; similarly for (n+1)L. Since the second sequence is the dual of the first it suffices to construct the first sequence. First, however, we need to prove some results about holomorphic vector fields on projective space.

Let $q: \mathbb{C}^{n+1} \setminus \{0\} \to P^n(\mathbb{C})$ denote the quotient map. Suppose X is a vector field on $\mathbb{C}^{n+1} \setminus \{0\}$ which is homogeneous of degree 1: $X(\lambda v) = \lambda X(v)$, $\lambda \in \mathbb{C}^*$, $v \in \mathbb{C}^{n+1} \setminus \{0\}$. Then q_*X is well-defined as a vector field on $P^n(\mathbb{C})$. Indeed, since $q = q.\lambda$, $Dq_v = Dq_{\lambda v}.\lambda$, $\lambda \in \mathbb{C}^*$. Hence $(q_*X)(q(v)) = Dq_v(X(v)) = Dq_{\lambda v}X(\lambda v)$, $v \in \mathbb{C}^{n+1} \setminus \{0\}$, $\lambda \in \mathbb{C}^*$, proving that q_*X is well-defined on $P^n(\mathbb{C})$. We define the *Euler vector field* E on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$E(z_0,\dots,z_n) = \sum_{i=0}^{n} z_i \,\partial/\partial z_i .$$

Certainly E is homogeneous of degree 1 and so q_*E is a well-defined holomorphic vector field on $P^n(\mathbb{C})$. Since $E(z) \in T_zL_z$, where L_z denotes the line through z and 0, we see that $q_*E = 0$. Let $\{s_0,\ldots,s_n\}$ be a basis of $\Omega(H)$. Since $\{s_0(z),\ldots,s_n(z)\}$ spans H_z for all $z \in P^n(\mathbb{C})$, to define a vector bundle morphism $(n+1)H \to TP^n$ it is enough to specify the map on holomorphic sections of $(n+1)H$. So suppose $(\phi_0,\ldots,\phi_n) \in \Omega((n+1)H)$. We define $\xi(\phi_0,\ldots,\phi_n) \in \Omega(TP^n)$ by

$$\xi(\phi_0,\ldots,\phi_n)(z) = \sum_{i=0}^{n} q_*(\phi_i(z)\partial/\partial z_i),$$

where we have used the identification between sections of H and linear functionals on $\mathbb{C}^{n+1}$. Now the kernel of ξ is the image of the map

$$T: \underline{\mathbb{C}} \to (n+1)H$$

defined on sections by $T(1)(z) = (z_0,\ldots,z_n)$, where $z_0,\ldots,z_n$ denote the coordinate functionals. In view of the fact that $T(1) = q_*E$, we see that the "Euler sequence"

$$0 \to \underline{\mathbb{C}} \xrightarrow{T} (n+1)H \xrightarrow{\xi} TP^n \to 0$$

is exact.

Taking the highest exterior power of the Euler sequence and using the result of Exercise 5, §1, we see that

$$\Lambda^n TP^n \approx \Lambda^{n+1}((n+1)H) \approx H^{n+1}.$$

Taking duals, we deduce that

$$\Lambda^n TP^{n*} \approx H^{-n-1}.$$

In the sequel we call the nth. exterior power of the cotangent bundle of an n-dimensional complex manifold M the *canonical bundle* of M and denote it by K(M). Thus we have shown that $K(P^n(\mathbb{C})) = H^{-n-1}$. As we shall see later the canonical bundle plays a central rôle in the theory of compact complex manifolds.

If X is a closed complex submanifold of $P^n(\mathbb{C})$, we may pull back (restrict) the hyperplane section bundle of $P^n(\mathbb{C})$ to X. We denote the resulting bundle on X by H_X. We remark that the zero sets of holomorphic sections of H_X are intersections of X with hyperplanes in $P^n(\mathbb{C})$. We refer to H_X as the *hyperplane section bundle of* X.

Divisors, holomorphic line bundles and linear systems.

Recall from §6 of Chapter 4 that the group $\mathcal{D}(M)$ of divisors on a complex manifold M is the set of (locally finite) formal sums

$\sum_{\alpha\in\Lambda} n_\alpha . V_\alpha$, where the n_α are integers and the V_α are irreducible analytic hypersurfaces of M. Here we suppose that M is compact and so we may assume that Λ is finite. By Theorem 4.6.11 every divisor on M may be specified by a Cartier divisor $\{(U_i,d_i): i \in I\} =$ $\{d_i \in \mathcal{M}^*(U_i): d_i d_j^{-1} \in A^*(U_{ij})\}$ and two Cartier divisors $\{(U_i,d_i): i \in I\}$ and $\{(V_j,e_j): j \in J\}$ determine the same element of $\mathcal{D}(M)$ if and only if $d_i e_j^{-1} \in A^*(U_i \cap V_j)$ for all $i \in I$, $j \in J$.

Before stating the next proposition we recall from Chapter 1 that the set HLB(M) of isomorphism classes of holomorphic line bundles on a complex manifold M has the natural structure of an Abelian group with composition defined by tensor product and inverse by dual. As in Chapter 1, we shall use the abbreviated notations E.F and E^{-1} for $E \otimes F$ and E* respectively. As usual $\underline{\mathbb{C}}$ will denote the trivial holomorphic line bundle.

Proposition 5.9.3. There is a canonical group homomorphism

$$[\ ]: \mathcal{D}(M) \to HLB(M)$$

Proof. Let $d = \{(U_i,d_i): i \in I\} \in \mathcal{D}(M)$. We let [d] denote the holomorphic line bundle on M with tranaition functions $\theta_{ij}: U_{ij} \to GL(1,\mathbb{C}) \approx \mathbb{C}^*$ defined by $\theta_{ij} = d_i/d_j$. We must show that [d] depends only on d and not on our particular representation of d as a Cartier divisor. Suppose then that $\{(U_i,d_i'): i \in I\}$ also defines the divisor d. The corresponding transition functions are given by $\theta_{ij}' = d_i'/d_j'$. But now $d_i/d_i' \in A^*(U_i)$ and so, setting $a_i = d_i/d_i'$, we have

$$a_i\theta_{ij} = \theta'_{ij}a_j, \quad i,j \in I.$$

Hence θ_{ij}, θ'_{ij} define isomorphic holomorphic line bundles (see Chapter 1, §5). The fact that [ ] is a group homomorphism is immediate from our definition of [ ] using Cartier divisors. □

Remark. As a consequence of Proposition 5.9.3 we see that if $d,d' \in \mathcal{D}(M)$, then $[d+d'] = [d][d']$ and $[d]^* = [-d]$.

Proposition 5.9.4. The sequence

$$\mathcal{M}^*(M) \xrightarrow{\text{div}} \mathcal{D}(M) \xrightarrow{[\ ]} \text{HLB}(M)$$

is exact.

Proof. Let $d = \{(U_i,d_i): i \in I\} \in \mathcal{D}(M)$ and suppose that $[d] = \underline{\mathbb{C}}$. Then there exist $a_i \in A^*(U_i)$ such that $a_i\theta_{ij} = a_j$, where $\theta_{ij} = d_i/d_j$ are the transition functions for $[d]$. Hence $a_i d_i/d_j = a_j$ and so $a_i d_i = a_j d_j$ on U_{ij}. Therefore we may define $m \in \mathcal{M}^*(M)$ by $m|U_i = a_i d_i$. Clearly $\text{div}(m) = d$ since $\text{div}(a_i d_i) = \text{div}(d_i)$, $i \in I$. Obviously $[\text{div}(m)] = \underline{\mathbb{C}}$ for all $m \in \mathcal{M}^*(M)$ and so we have shown that $\text{div}(\mathcal{M}^*(M)) = \text{Kernel}[\ ]$. □

Definition 5.9.5. Let $d,d' \in \mathcal{D}(M)$. We say that d and d' are *linearly equivalent* if $d-d'$ is the divisor of a meromorphic function. We denote the group of linear equivalence classes of divisors on M by $L(M)$. Thus $L(M) = \mathcal{D}(M)/\text{div}(\mathcal{M}^*(M))$. Given $d \in \mathcal{D}(D)$, we let $L(d)$ denote the set of all divisors on M linearly equivalent to d.

Next we wish to define *meromorphic sections* of a holomorphic line bundle. Suppose that $E \in \text{HLB}(M)$ has transition functions $\theta_{ij}: U_{ij} \to \mathbb{C}^{\bullet}$. We say that a family $m_i \in \mathcal{M}(U_i)$, $i \in I$, defines a meromorphic section of E if $\theta_{ij}m_j = m_i \in \mathcal{M}(U_{ij})$, $i,j \in I$. We let $\mathcal{M}(E)$ denote the space of meromorphic sections of E and $\mathcal{M}^*(E)$ denote the space of non-zero meromorphic sections of E.

Remark. The group law in HLB(M) induces corresponding maps on spaces of sections. For example, if $E \in \text{HLB}(M)$ and $s \in \mathcal{M}^*(E)$ has local representatives $s_i \in \mathcal{M}^*(U_i)$, we may define $s^{-1} \in \mathcal{M}^*(E^{-1})$ to be the section with local representatives $(s^{-1})_i = s_i^{-1}$. This construction defines an inversion map $\mathcal{M}^*(E) \to \mathcal{M}^*(E^{-1})$. Similarly if $E,F \in \text{HLB}(M)$ and

$E,F \in HLB(M)$ and $(s,t) \in M(E) \times M(F)$ we may define the composition $s.t \in M(E.F)$ by $s.t = s \otimes t$.

Proposition 5.9.6. Let $E \in HLB(M)$ and suppose that $M^*(E) \neq \emptyset$. We have a natural group homomorphism div: $M^*(E) \to \mathcal{D}(M)$ satisfying

$$[\mathrm{div}(s)] = E, \ s \in M^*(E).$$

Proof. Let E have transition functions $\theta_{ij}: U_{ij} \to \mathbb{C}^{\bullet}$ and $s \in M^*(E)$ have corresponding local representatives $s_i \in M^*(U_i)$. Since $\theta_{ij}s_j = s_i$, we have $s_i s_j^{-1} \in A^*(U_{ij})$ and so we may define div(s) to be the Cartier divisor $\{(U_i, s_i): i \in I\}$. Clearly div(s) depends only on s and not on our particular choice of transition functions for E. It is immediate from our local description of div(s) that $[\mathrm{div}(s)] = E$. Finally, div: $M^*(E) \to \mathcal{D}(M)$ is obviously a group homomorphism. □

Remark. We call div(s) the *divisor* of the section s.

Example 1. Let $m \in M^*(P^n(\mathbb{C}))$. Then $\deg(\mathrm{div}(m)) = 0$ (Example 3, §6, Chapter 4). Suppose that E is a holomorphic line bundle on $P^n(\mathbb{C})$ and $s,t \in M^*(E)$. Since $st^{-1} \in M^*(P^n(\mathbb{C}))$, we see at once that $\deg(\mathrm{div}(s)) = \deg(\mathrm{div}(t))$. Hence we may define the degree of E, deg(E), to be the degree of any non-trivial meromorphic section of E. Clearly, deg: $HLB(P^n(\mathbb{C})) \to \mathbb{Z}$ is a homomorphism. See also Proposition 1.5.7. We shall give another interpretation of the degree map in §3, Chapter 6.

Proposition 5.9.7. Let $d \in \mathcal{D}(M)$. Then

1. There exists $s(d) \in M^*([d])$ such that $\mathrm{div}(s(d)) = d$. The section s(d) is unique up to multiplication by elements of $\mathbb{C}^{\bullet}$.

2. $\mathrm{div}(M^*([d])) = L(d)$. Moreover the map div: $M^*([d]) \to L(d)$ induces a bijection of $L(d)$ with $M^*([d])/\mathbb{C}^{\bullet}$.

Proof. The proof of 1 is the same as the proof of Proposition 1.5.4. Let us prove part 2. Suppose that $s \in M^*([d])$. Then $[\mathrm{div}(s)] \cong [d]$ and so $s(d)^{-1}s \in M^*(M)$. Therefore, $\mathrm{div}(s) \in L(d)$. Conversely, if $d' \in L(d)$, there exists $m \in M^*(M)$ such that $d-d' = \mathrm{div}(m)$. Hence $[d] \cong [d']$ and $s(d')$ determines a section of $[d]$ with divisor $d' = d-\mathrm{div}(m) \in L(d)$. The remaining assertion of part 2 is immediate from part 1. □

Proposition 5.9.8. A holomorphic line bundle E lies in the image of $[\ ]: \mathcal{D}(M) \to HLB(M)$ if and only if E has a non-trivial meromorphic section.

Proof. Immediate from proposition 5.9.6 and 5.9.7. □

Remark. Proposition 5.9.8 shows that the study of divisors on M is closely related to the problem of finding which holomorphic line bundles on a complex manifold admit non-trivial meromorphic sections. Two fundamental results that we prove later show that if M is projective or Stein then every holomorphic vector bundle on M admits a non-trivial meromorphic section. It must be stressed that an arbitrary complex manifold of dimension greater than 1 need not have any holomorphic line bundles which admit meromorphic sections (equivalently, the manifold need not have any divisors).

Definition 5.9.9. We say that a divisor $d = \sum_{\alpha \in \Lambda} n_\alpha . V_\alpha$ is *effective* or *positive* if $n_\alpha \geq 0$, $\alpha \in \Lambda$. We write $d \geq 0$.

Suppose that $d \in \mathcal{D}(M)$. We let L(d) denote the vector subspace of $\mathcal{M}(M)$ defined by

$$m \in L(d) \text{ iff } d + \operatorname{div}(m) \geq 0 \text{ or } m = 0.$$

Proposition 5.9.10. The vector space L(d) is isomorphic to $\Omega([d])$. In particular $\dim_{\mathbb{C}} L(d) < \infty$.

Proof. By Proposition 5.9.7, there exists $s \in \mathcal{M}^*([d])$ such that $\operatorname{div}(s) = d$. Let $\gamma: \mathcal{M}(M) \to \mathcal{M}([d])$ denote the map defined by $\gamma(m) = s.m$, $m \in \mathcal{M}(M)$. Now $\operatorname{div}(\gamma(m)) = \operatorname{div}(s) + \operatorname{div}(m)$ and so if $m \in L(d)$ we see that $\operatorname{div}(\gamma(m)) \geq 0$. That is, $\gamma(m) \in \Omega([d])$. The map γ clearly restricts to a linear isomorphism between L(d) and $\Omega([d])$ with inverse defined by $\gamma^{-1}(m) = s^{-1}.m$, $m \in \Omega([d])$. □

Remarks.

1. It follows from Proposition 5.9.7, part 1, that the isomorphism between L(d) and $\Omega([d])$ is uniquely determined by d up to scalar multiplication by elements of $\mathbb{C}^{\bullet}$.

2. It follows from Proposition 5.9.10 that every non-zero meromorphic function on M can be expressed as a quotient of holomorphic sections of some holomorphic line bundle on M. Indeed, if $m \in \mathcal{M}^*(M)$ let $P_m = \min(0, \mathrm{div}(m))$ denote the polar divisor of m and choose $s \in \Omega([-P_m])$ such that $\mathrm{div}(s) = -P_m$. By Proposition 5.9.10 the map $\gamma^{-1}: \Omega([-P_m]) \to L(-P_m)$ defined by $\gamma^{-1}(t) = t/s$ is an isomorphism. But $m \in L(-P_m)$ and so m can be written as a quotient of holomorphic sections of the line bundle $[-P_m]$.

Example 2. Let $m \in \mathcal{M}^*(P^n(\mathbb{C}))$. Assuming Chow's theorem, we may write m as a quotient P/Q, where P and Q are homogeneous polynomials of the same degree. If the common degree of P and Q is d, we see from Proposition 5.9.2, that m is the quotient of the holomorphic sections of H^d determined by P and Q.

Given $d \in \mathcal{D}(M)$, we let $E(d)$ denote the set of all effective divisors linearly equivalent to d. The map $\mathrm{div}: \Omega([d]) \to E(d)$ induces an isomorphism between $E(d)$ and $P(\Omega([d]))$. Hence for every divisor d on M we have natural isomorphisms

$$E(d) \approx P(L(d)) \approx P(\Omega([d])).$$

A family Σ of effective divisors on M is called a *linear system of divisors* on M if there exists a holomorphic line bundle E on M and a (projective) linear subspace V of $P(\Omega(E))$ such that $\Sigma = \mathrm{div}(V)$. We say that Σ is a *complete linear system* if $\Sigma = \mathrm{div}(P(\Omega(E)))$ for some $E \in HLB(M)$.

If Σ is the linear system of divisors on M corresponding to the subspace V of $P(\Omega(E))$, we define the dimension of Σ, $\dim(\Sigma)$, to be $\dim_{\mathbb{C}}(V)$. We see that if $\dim_{\mathbb{C}}(V) = n$, then Σ is parametrized by $P^n(\mathbb{C})$ and we may write $\Sigma = \{d_t : t \in P^n(\mathbb{C})\}$.

Let $d \in \mathcal{D}(M)$. If V is a linear subspace of $P(\Omega([d]))$, then $\mathrm{div}(V)$ is a linear system on M. In case $V = P(\Omega([d]))$, we see that $\dim(\Sigma) = \dim_{\mathbb{C}}\Omega([d])) - 1 = \dim_{\mathbb{C}}L(d) - 1$ and the linear system equals $E(d)$. By the remarks above it is clear that linear systems on M always correspond to a subspace of $E(d)$ for some divisor d on M and that a linear system is complete if and only if it equals $E(d)$ for some divisor d on M.

Suppose $\Sigma = \{d_t : t \in P^n(\mathbb{C})\}$ is a linear system on M. We define the *base locus* B_Σ of Σ to be the analytic set

$$B_\Sigma = \bigcap_{t \in P^n(\mathbb{C})} |d_t| .$$

Let $t_0, \dots, t_n$ be linearly independent in $P^n(\mathbb{C})$. Then

$$B_\Sigma = \bigcap_{j=0}^{n} |d_{t_j}|.$$

Since Σ is a linear system, there exists $L \in HLB(M)$ and a linear subspace V of $P(\Omega(L))$ such that $\Sigma = div(V)$. Let $\{s^0, \dots, s^n\}$ be a basis for V then we clearly have

$$B_\Sigma = \bigcap_{j=0}^{n} (s^j)^{-1}(0).$$

Suppose $B_\Sigma = \emptyset$. Then for each $x \in M$, the (n+1)-tuple $(s^0(x), \dots, s^n(x))$ defines a unique point in $P^n(\mathbb{C})$. Indeed, if $s_i^0, \dots, s_i^n$ and $s_j^0, \dots, s_j^n$ are local representations of $s^0, \dots, s^n$ with respect to trivialisaions over U_i and U_j respectively, we see that $s_i^p(x) = \theta_{ij}(x) s_j^p(x)$, $0 \le p \le n$, $x \in U_{ij}$, where $\theta_{ij} \in A^*(U_{ij})$. Since $\theta_{ij}(x) \neq 0$, $(s_i^0(x), \dots, s_i^n(x)) = (s_j^0(x), \dots, s_j^n(x)) \in P^n(\mathbb{C})$. Hence provided $B_\Sigma = \emptyset$, we have a holomorphic map $P: M \to P^n(\mathbb{C})$ defined by $P(x) = (s^0(x), \dots, s^n(x))$, $x \in M$.

It is of great interest to know whether a given holomorphic line bundle L on M has "enough" holomorphic sections to determine an *embedding* of M in projective space. Notice that if X is a complex submanifold of $P^n(\mathbb{C})$, H_X denotes the hyperplane section bundle of $P^n(\mathbb{C})$ restricted to X and Σ denotes the complete linear system corresponding to $\Omega(H_X)$, then $B_\Sigma = \emptyset$ and the corresponding map $P: X \to P^n(\mathbb{C})$ is an embedding onto the submanifold X.

In conclusion, we see that the theory of divisors and meromorphic functions on a compact complex manifold is intimately related with the theory of holomorphic line bundles and their holomorphic and meromorphic sections. We mention the following basic problems:

1. The existence of non-trivial meromorphic sections of a given holomorphic line bundle.

2. Relations between the dimension of the space of holomorphic sections of a holomorphic line bundle L on M and other invariants of L and M.

3. Conditions for the existence of sufficiently many holomorphic sections of a holomorphic line bundle L on M for it to determine an embedding of M in projective space.

Geometric genus.

Let M be a compact complex manifold of dimension m and let K(M) denote the canonical bundle $\Lambda^m TM^*$ of M. We define the *geometric genus* $p_g(M)$ of M to be $\dim_{\mathbb{C}}(\Omega(K(M)))$.

The geometric genus is obviously a biholomorphic invariant.

Proposition 5.9.11. The geometric genus is invariant under blowing ups with non-singular centres.

Proof. Let $\pi: \tilde{M} \to M$ denote the blow-up of M with centre p and exceptional variety E. We have an induced map $\pi^*: \Omega(K(M)) \to \Omega(K(\tilde{M}))$. Since π restricts to a biholomorphic map between $M \setminus E$ and $M \setminus \{p\}$ we see by uniqueness of analytic continuation that π^* is injective. On the other hand if $\phi \in \Omega(K(\tilde{M}))$, we may define $\tilde{\phi} = (\pi | M \setminus E)_* \phi \in K(M \setminus \{p\})$. By Hartog's theorem, $\tilde{\phi}$ extends to a holomorphic section of K(M) which we denote by $\pi_*\phi$. Clearly $\pi^*\pi_*\phi = \phi$, $\phi \in K(\Omega(\tilde{M}))$ and so π^* is a linear isomorphism. Hence $p_g(M) = p_g(\tilde{M})$. The proof for general non-singular centres is similar, using the second Riemann removable singularities theorem (Exercise 5, §2, Chapter 4) and we leave details to the reader. □

Remark. The proof given for proposition 5.9.11 also shows that the numbers $\dim_{\mathbb{C}}\Omega(\Lambda^p TM^*)$, $p > 0$, are also invariant under blowings up with non-singular centres.

The geometric genus is in fact a *bimeromorphic* invariant. Whilst we shall not define bimeromorphic maps here (see Ueno [1] for details and references) we point out that bimeromorphic maps are the complex analytic analogue of the birational maps of algebraic geometry. Moreover, if two complex analytic *surfaces* are bimeromorphic then one is obtained from the other by a finite sequence of blow ups and blow downs (see Kodaira [1]). This result is, however, false in higher dimensions. We refer to Hartshorne [1 ; pages 412-414] for a discussion of the higher dimensional case and references. Other bimeromorphic invariants related to the geometric genus are the *plurigenera* $p_r(M)$ defined by $p_r(M) = \dim_{\mathbb{C}}\Omega(\otimes^r K(M))$, $r > 0$. The proof that the plurigenera are invariant under blowing ups is similar to that of Proposition 5.9.11.

Holomorphic line bundles on complex tori and theta functions.

We conclude this section by indicating the role of holomorphic line bundles and their sections in the study of meromorphic functions on complex tori. For further details, proofs and references the reader may consult Cornalba [1], Griffiths and Harris [1], Swinnerton-Dyer [1] and Weil [1].

Let $T = \mathbb{C}^n/\Lambda$ be an n-dimensional complex torus with period lattice Λ and $\pi: \mathbb{C}^n \to T$ denote the quotient map. If L is a holomorphic line bundle on T then π^*L is a holomorphic line bundle on $\mathbb{C}^n$ (As usual, π^*L denotes the pull-back of the bundle L by π - see the exercises at the end of §5, Chapter 1). By Corollary 5.8.4, every holomorphic line bundle on $\mathbb{C}^n$ is holomorphically trivial and so $\pi^*L \cong \underline{\mathbb{C}} = \mathbb{C}^n \times \mathbb{C}$. Fixing an isomorphism of π^*L with $\underline{\mathbb{C}}$, we may regard L as the quotient of $\mathbb{C}^n \times \mathbb{C}$ under the identifications

$$(z,v) \sim (z+\lambda, f_\lambda(z)v),\quad z \in \mathbb{C}^n,\ v \in \mathbb{C},\ \lambda \in \Lambda,$$

where the functions $f_\lambda \in A^*(\mathbb{C}^n)$ satisfy the relations

$$f_\lambda(z+\mu)f_\mu(z) = f_{\lambda+\mu}(z),\quad z \in \mathbb{C}^n,\ \lambda,\mu \in \Lambda \qquad \ldots.(A)$$

A (holomorphic) section of L corresponds to a $\mathbb{C}$-valued (holomorphic) function θ on $\mathbb{C}^n$ which satisfies the functional equation

$$\theta(z+\lambda) = f_\lambda(z)\theta(z),\quad z \in \mathbb{C}^n,\ \lambda \in \Lambda.$$

Such functions are called *theta functions* (relative to the family f_λ). The quotient of any two non-zero theta functions defines a Λ-periodic meromorphic function on $\mathbb{C}^n$ and hence a meromorphic function on T.

Conversely, by Remark 2 following Proposition 5.9.10, every meromorphic function on T may be represented as the quotient of two theta functions (associated to the same family f_λ). The study of meromorphic functions on T may therefore be reduced to the study of theta functions on $\mathbb{C}^n$. In fact for n > 1, this approach to the theory of meromorphic functions on complex tori is much more effective than any direct attempt to construct meromorphic functions as we did in case n = 1 with the Weierstrass $\wp$-function and its derivative.

Suppose that α, β are isomorphism of π^*L with $\underline{\mathbb{C}}$. Then there exists $\phi \in A^*(\mathbb{C}^n)$ such that $\beta = \phi.\alpha$. If α, β correspond to the families $\{f_\lambda: \lambda \in \Lambda\}$, $\{g_\lambda: \lambda \in \Lambda\}$ respectively, then it is easily verified that f_λ and g_λ are related by

$$g_\lambda(z) = \phi(z+\lambda)\phi(z)^{-1}f_\lambda(z), \ z \in \mathbb{C}^n, \ \lambda \in \Lambda.$$

By suitable choice of ϕ we might hope to put the functions f_λ into a "standard" form. This amounts to obtaining a classification of HLB(T).

Example 3. The trivial line bundle on T is defined by the family $f_\lambda \equiv 1$, $\lambda \in \Lambda$. Given $\phi \in A^*(\mathbb{C}^n)$, we may write $\phi = \exp(f)$, for some $f \in A(\mathbb{C}^n)$. The family $g_\lambda(z) = \exp(f(z+\lambda) - f(z))$ also defines the trivial line bundle on T. Since $\phi(z+\lambda) = g_\lambda(z)\phi(z)$, we see that ϕ is a theta function for this family. Any theta function of this type is called a *trivial theta function* by virtue of the fact that it corresponds to a constant (non-zero) section of the trivial holomorphic line bundle on T. Of special interest to us will be the case when $f(z) = az^2+bz+c, a,b,c \in \mathbb{C}$. We have $g_\lambda(z) = \exp(2az + z\lambda^2 + b\lambda)$ and the corresponding trivial theta function is $\exp(az^2 + bz + c)$.

If L is a holomorphic line bundle on T it is natural to try to define L by functions f_λ where $f_\lambda(z) = \exp F(z,\lambda)$ and $F(z,\lambda)$ is affine linear in z. Suppose L is defined by a family of functions of this type. The relations (A) impose conditions on the $F(z,\lambda)$ and, multiplying by suitable non-zero analytic functions, it is not hard to show that the f_λ can be put in the form

$$f_\lambda(z) = m(\lambda)\exp(\pi H(z,\lambda) + \tfrac{1}{2}\pi H(\lambda,\lambda)) ,$$

where H is an Hermitian form whose imaginary part E is integer valued on $\Lambda \times \Lambda$ and $m: \Lambda \to S^1 \subset \mathbb{C}$ satisfies $m(\lambda)m(\mu) = m(\lambda+\mu)\exp(\pi i E(\lambda,\mu))$ for all $\lambda, \mu \in \Lambda$.

We denote the line bundle on T corresponding to H and m by L(H,m). By a theorem of Apell and Humbert every holomorphic line bundle on T is isomorphic to a line bundle of the form L(H,m). Moreover, $L(H_1,m_1) \cong L(H_2,m_2)$ iff $H_1 = H_2$ and $m_1 = m_2$. The proof of this result may be found in the references. Here we only remark that $L(H_1,m_1)$ and $L(H_2,m_2)$ are isomorphic as *complex* line bundles iff $H_1 = H_2$.

A holomorphic section of L(H,m) corresponds to a theta function θ which satisfies the functional equation

$$\theta(z+\lambda) = m(\lambda)\exp(\pi H(z,\lambda) + \tfrac{1}{2}\pi H(\lambda,\lambda))\theta(z) .$$

Example 4. (The Weierstrass σ-function). We follow the notation and assumptions of §4 of Chapter 4. Thus we assume n = 1 and let $\wp(z)$ denote the Weierstrass elliptic function associated to the lattice L generated by $\{\omega_1, \omega_2\}$. Integrating $\wp$ we obtain the Weierstrass zeta function

$$\zeta(z) = z^{-1} - \sum{}'[(z+\omega)^{-1} - \omega^{-1} + 2\omega^2]$$

(the prime denotes that the sum is over non-zero elements of L). $\zeta(z+\omega) - \zeta(z)$ is constant, not necessarily zero, for all $\omega \in L$ and we define

$$\eta_1 = \zeta(z+\omega_1) - \zeta(z); \ \eta_2 = \zeta(z+\omega_2) - \zeta(z) \qquad \ldots.(B)$$

As in the proof of part 3 of Theorem 4.4.2, the integral of ζ round a period parallelogram for L equals $2\pi i = \omega_1\eta_2 - \omega_2\eta_1$ (Legendre's relation).

Exponentiating the integral of ζ we obtain a (single valued) analytic function called the Weierstrass σ-function. We have

$$\begin{aligned}\sigma(z) &= \exp(\log z + \int_0^z (\zeta(t) - t^{-1})dt) \\ &= z\prod{}' \left(\frac{z+\omega}{\omega} \exp\left(-\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)\right)\end{aligned}$$

Exponentiating the integrals of the relations (B) above, substituting $z = -\frac{1}{2}\omega_1, -\frac{1}{2}\omega_2$ and using the fact that σ is an odd function which does not vanish at $\frac{1}{2}\omega_1, \frac{1}{2}\omega_2$, we find that

$$\sigma(z+\omega) = (-1)^n\sigma(z)\exp(\eta(z+\tfrac{1}{2}\omega)),$$

where $\omega = n_1\omega_1 + n_2\omega_2$, $\eta = n_1\eta_1 + n_2\eta_2$, $n = n_1n_2 + n_1 + n_2$ and $n_1, n_2 \in \mathbb{Z}$. Hence σ is a theta function on $\mathbb{C}$ corresponding to the family $\{g_\omega \in A^*(\mathbb{C}) : \omega \in L\}$ defined by

$$g_\omega(z) = (-1)^n\exp(\eta(z+\tfrac{1}{2}\omega)), \ \omega \in L.$$

However, the functions g_ω are not in the standard form that we gave above. Recall from §4 of Chapter 4 that the lattice L has Riemann form defined by $A(y,z) = S^{-1}\mathrm{Im}(y\bar{z})$, where $S = \mathrm{Im}(\omega_1\bar{\omega}_2)$. Associated to A we have the hermitian form H defined by $H(y,z) = y\bar{z}/S$. The imaginary part of H equals A and is integer valued on $L \times L$. We now define

$$f_\omega(z) = (-1)^n\exp(\pi H(z,\omega) + \tfrac{1}{2}\pi H(\omega,\omega)), \ \omega \in L.$$

Set $a = (2\omega_1)^{-1}(S^{-1}\pi\bar{\omega}_1 - \eta_1)$ and let $\theta_0(z)$ denote the trivial theta function $\exp(az^2)$. The reader may verify, using Legendre's relation, that if we define $\tilde{\sigma}(z) = \theta_0(z)\sigma(z)$, then $\tilde{\sigma}(z+\omega) = f_\omega(z)\tilde{\sigma}(z)$. Hence we have put the Weierstrass σ-function in standard form. Set $T = \mathbb{C}/L$. Because the holomorphic line bundle on T associated to the family f_ω actually generates HLB(T) we are able to give a particularly simple description of meromorphic functions on T. Thus if $d = \sum_{k=1}^{n} n_k \cdot z_k \in \mathcal{D}(T)$, $\deg(d) = 0$ and $\sum_{k=1}^{n} n_k z_k = e$, we may define $m_d = \prod_{k=1}^{n} \sigma(z-a_k)^{n_k}$, where $\pi(a_k) = z_k$, $k = 1,\ldots,n$, and $\sum_{k=1}^{n} n_k a_k = 0$. It is easily verified that m_d is L-elliptic and defines a meromorphic function on T with divisor d.

The theory of theta functions of more than 1 complex variable is highly developed and we conclude by mentioning just two important results:

1. If there exists $L(H,m) \in HLB(T)$ such that H is positive definite (that is, H is associated to a Riemann form), then the complete linear system defined by $L(3H,m) = L(H,m)^3$ gives a projective embedding of T.

2. The dimension of $\Omega(L(H,m))$ can be computed and is equal to $\sqrt{\det E}$, where E is the imaginary part of H and the determinant is computed relative to any basis of the period lattice of T.

Exercises.

1. Let s be a meromorphic section of the holomorphic line bundle E. Define the zero and pole sets Z(s), P(s) of s and show that s is holomorphic if and only if $P(x) = \emptyset$.

2. Let $p \geq 1$ and Σ denote the complete linear system on $P^n(\mathbb{C})$ corresponding to $\Omega(H^p)$. Show that

a) $\dim(\Sigma) = \binom{n+p}{n} - 1$.

b) The base locus of Σ is empty.

c) Σ determines an embedding of $P^n(\mathbb{C})$ in $P^N(\mathbb{C})$, $N = \dim(\Sigma)$ (The "p-tuple embedding").

(In case $n = 2$, $p = 2$, we obtain an embedding of $P^2(\mathbb{C})$ in $P^5(\mathbb{C})$. The corresponding surface in $P^5(\mathbb{C})$ is called the *Veronese surface*).

3. Let E be a holomorphic line bundle on the complex manifold M. Show that, as vector spaces, $M(E) \cong M(M)$, provided that E admits a non-trivial meromorphic section.

4. Suppose V is a smooth analytic hypersurface in the compact complex manifold M (that is, V is a closed submanifold of M of codimension 1). Let $N_V = (TM|V)/TV$ denote the normal bundle of V. Show that

a) $N_V^* \approx \{v \in TM^*|V: v \text{ is zero on } TV^*\}$.

b) $N_V^* \cong [-V]|V$.

c) $K(V) \cong (K(M) \otimes [V])|V$.

(Hints: For b), show that the local defining equations for V determine a non-zero holomorphic section of $N_V^* \otimes [V]|V$; for c), use the exact sequence $0 \to N_V^* \to TM^*|V \to TV^* \to 0$ and exercise 5, §1).

5. Suppose that $f_\lambda \in A^*(\mathbb{C}^n)$, $\lambda \in \Lambda$, define the holomorphic line bundle L on $\mathbb{C}^n/\Lambda$. Show that f_λ^{-1}, $\bar{f}_\lambda$, $\bar{f}_\lambda^{-1}$ respectively define the complex line bundles L^*, $\bar{L}$, $\bar{L}^*$ on $\mathbb{C}^n/\Lambda$.

6. Let Λ be a lattice in $\mathbb{C}^n$ and H be an hermitian form whose imaginary part is integer valued on $\Lambda \times \Lambda$. Show that L(H,1) is a holomorphic line bundle on $\mathbb{C}^n/\Lambda$. Now set L = L(H,1). Show that a function $\theta: \mathbb{C}^n \to \mathbb{C}$ determines a section of $L^* \otimes \bar{L}^*$ if and only if for all $z \in \mathbb{C}^n$ we have

$$\theta(z+\lambda) = \exp(-\pi(2\mathrm{Re}(H(z,\lambda)) + H(\lambda,\lambda))\theta(z), \ \lambda \in \Lambda.$$

Deduce that $\phi(z) = \exp(-\pi H(z,z))$ determines a nowhere vanishing smooth section of $L^* \otimes \bar{L}^*$ (We may think of ϕ as determining a canonical hermitian form on L = L(H,1)).

§10. Pseudoconvexivity and Stein manifolds.

In this section we shall show how some of the pseudoconvexivity definitions we discussed in Chapter 2 may be generalised to arbitrary non-compact complex manifolds.

We start with a few remarks about Hermitian forms on an m-dimensional complex vector space E. Recall that $H: E \times E \to \mathbb{C}$ is said to be an *Hermitian form* if

1. $H(x,y) = \overline{H(y,x)}$, $x,y \in E$.

2. $H(ax_1+bx_2,y) = aH(x_1,y) + bH(x_2,y)$, $a,b \in \mathbb{C}$, $x_1,x_2,y \in E$.

We say that H is *positive definite* if, in addition,

3. $H(x,x) > 0$, $x \neq 0$.

Conditions 1 and 2 imply that H is conjugate complex linear in the second variable. Consequently, an Hermitian form may be regarded as a complex bilinear map $H: E \times \bar{E} \to \mathbb{C}$ satisfying the conjugate symmetry condition 1. Since the space of complex bilinear maps of $E \times \bar{E}$ to $\mathbb{C}$ is naturally isomorphic to $E^* \otimes \bar{E}^*$, we may also regard H as lying in $E^* \otimes \bar{E}^*$. Thus, relative to a basis of E, we may write H in coordinates as

$$H = \sum_{i,j=1}^{m} h_{i\bar{j}} dz_i \otimes d\bar{z}_j.$$

The conjugate symmetry condition amounts to requiring that the matrix $[h_{i\bar{j}}]$ be Hermitian. That is, $\overline{h_{i\bar{j}}} = h_{j\bar{i}}$, $1 \leq i, j \leq m$.

Next observe that $E^* \otimes \bar{E}^* \approx \Lambda^{1,1}(E')$. If we regard H as lying in $\Lambda^{1,1}(E')$, the conjugate symmetry condition amounts to requiring that $\bar{H} = -H$ (conjugate the form $\Sigma h_{i\bar{j}} dz_i \wedge d\bar{z}_j$).

Finally note that since $E^* \otimes \bar{E}^* \approx L(E, \bar{E}^*)$, we may regard H as an element of $L(E, \bar{E}^*)$. In this case conjugate symmetry amounts to $H = \bar{H}^*$.

Recall that if we choose a complex basis for E and let H have matrix $[h_{i\bar{j}}]$ relative to this basis, then the integers

$n(H)$ = number of negative eigenvalues of $[h_{i\bar{j}}]$,
$z(H)$ = number of zero eigenvalues of $[h_{i\bar{j}}]$,
$p(H)$ = number of positive eigenvalues of $[h_{i\bar{j}}]$,

are invariants of H which do not depend on the choice of basis for E.

Definition 5.10.1. Let M be a complex manifold. An Hermitian form on M is a section H of $TM^* \otimes \overline{TM}^*$ such that $H(x)$ is an Hermitian form on T_xM for all $x \in M$.

Remark. We may equivalently define an Hermitian form on M to be a (1,1)-form H satisfying $\bar{H} = -H$.

Definition 5.10.2. Let M be a complex manifold and $\phi \in C^2_{\mathbb{R}}(M)$. The *Levi form* of ϕ is the (1,1)-form defined by

$$L(\phi) = \partial\bar{\partial}\phi.$$

Since $\overline{\partial\bar{\partial}\phi} = \bar{\partial}\partial\phi = -\partial\bar{\partial}\phi$, we see that $L(\phi)$ is an Hermitian form on M. In local coordinates,

$$L(\phi) = \sum_{i,j=1}^{m} \partial^2\phi/\partial z_i \partial\bar{z}_j \; dz_i \otimes d\bar{z}_j.$$

and so $L(\phi)$ is an invariant version of the Levi form we discussed in Chapter 2.

Suppose now that M is a relatively compact domain in the m-dimensional complex manifold $\tilde{M}$ and that M has C^2 boundary ∂M. We say that $\phi \in C^2_{\mathbb{R}}(\tilde{M})$ is a *defining function* for M if

1. $M = \{x \in M: \phi(x) < 0\}$.
2. $\partial M = \phi^{-1}(0)$.
3. $d\phi \neq 0$ on ∂M.

Given a defining function ϕ for M we may define the *holomorphic tangent space* $T_x\partial M$ to ∂M at x by $T_x\partial M = \{v \in T_xM: d\phi(x)(v) = 0\}$. Set

$$T\partial M = \cup_{x\in\partial M} T_x\partial M.$$

It is straightforward to verify that $T\partial M$ is an $(m-1)$-dimensional complex vector bundle which is defined independently of choices of defining function for M (Use Lemmas 2.5.1, 2.5.7).

Given $x \in \partial M$, we set $L(\phi)(x) = L(\phi)|T_x\partial M$ and define

$$n(x) = n(L(\phi)(x))$$

$$z(x) = z(L(\phi)(x))$$

$$p(x) = p(L(\phi)(x)).$$

Clearly $n(x) + z(x) + p(x) = m-1$ and it is a straightforward exercise to verify that $n(x)$, $z(x)$ and $p(x)$ depend only on $x \in \partial M$ and not the choice of defining function ϕ (see §5, Chapter 2).

Definition 5.10.3. Let M be a relatively compact domain of the complex manifold $\tilde{M}$ and assume that M has C^2 boundary. Suppose that M has defining function ϕ. Then we say that M is *q-pseudoconvex* (respectively, *strictly* q-pseudoconvex) if for all $x \in \partial M$ we have

$$n(x) \leq q \text{ (respectively, } n(x) + z(x) \leq q).$$

Remark. As in Proposition 2.5.12, we may show that q-pseudoconvexity is a local property of the boundary.

Example 1. Let Ω be an L.p. (respectively, s.L.p.) domain in $\mathbb{C}^n$. Then Ω is 0-pseudoconvex (respectively, strictly 0-pseudoconvex).

Remark. In the sequel we shall always refer to 0-pseudoconvex (respectively, strictly 0-psuedoconvex) domains as L.p. (respectively, s.L.p.) domains.

Proposition 5.10.4. Let M be an s.L.p. domain in $\tilde{M}$. Then there exists a C^2 defining function ϕ for M such that $L(\phi)|\partial M$ is positive definite.

Proof. Same as the proofs of Propositions 2.5.5 and Lemma 2.5.11. □

In Chapter 7 we shall prove the basic result of Grauert to the effect that an s.L.p. domain is holomorphically convex. However, an s.L.p. domain need not be Stein.

Examples.

2. The unit Euclidean disc $E(1) \subset \mathbb{C}^n$ is s.L.p. (take $\phi(z) = \Sigma|z_i|^2 - 1$). Suppose $n > 1$ and let M, $\tilde{M}$ denote the result of blowing up $E(1)$, $\mathbb{C}^n$ at zero. Then M will be an s.L.p. domain in $\tilde{M}$ as we may choose a defining function for M which is equal to ϕ on a neighbourhood of ∂M in $\tilde{M}$ which does not contain the exceptional variety of the blowing up. However, M cannot be Stein as the exceptional variety of the blowing up is a compact complex submanifold of M biholomorphic to $P^{n-1}(\mathbb{C})$. In particular, we cannot separate points on $P^{n-1}(\mathbb{C})$ by holomorphic functions on M (Proposition 4.2.4).

3. Let L denote the universal line bundle on $P^n(\mathbb{C})$. Then L is biholomorphic to $\mathbb{C}^{n+1}$ blown up at zero. Hence, by example 2, there is an s.L.p. neighbourhood of the zero section of L. More generally, if X is any compact complex submanifold of $P^n(\mathbb{C})$ then $L_X = L|X$ has an s.L.p. neighbourhood of the zero section.

Motivated by example 2, we now give an important definition due to Grauert [1].

Definition 5.10.5. Let E be a holomorphic vector bundle on the compact complex manifold M. We say that E is *weakly negative* if there is an s.L.p. neighbourhood of the zero section of E. We say that E is *weakly positive* if E^* is weakly negative.

Remark. One of the main theorems we prove in Chapter 7 is that if a compact complex manifold M admits a weakly positive vector bundle then M is algebraic.

Our definition of the q-pseudoconvexivity of a complex manifold M depended on representing M as a domain in some larger complex manifold. Our next aim is to present an intrinsic definition of pseudoconvexivity.

Definition 5.10.6. Suppose that M is an m-dimensional complex manifold and let $\phi \in C^{\infty}_{\mathbb{R}}(M)$. We say that ϕ is *strictly q-plurisubharmonic* (abbreviated, strictly q-psh) if $L(\phi)(x)$ has at least $m-q$ positive eigenvalues at every point x of M.

Recall from §5, Chapter 2, that $\phi \in C^{\infty}_{\mathbb{R}}(M)$ is said to be an *exhaustion function* for M if for all $c \in \mathbb{R}$,

$$M_c = \{x \in M: \phi(x) < c\}$$

is relatively compact subset of M.

Definition 5.10.7. A complex manifold M is said to be *q-complete* if there exists a strictly q-psh exhaustion function ϕ on M.

Remarks.

1. Clearly q-complete implies (q+1)-complete.

2. We often refer to 0-complete manifolds as being *holomorphically complete.*

Theorem 5.10.8. Every Stein manifold is 0-complete.

Proof. Exactly the same as the proof of Theorem 2.5.20. □

Remark. We shall prove in Chapter 11 that every 0-complete manifold is Stein.

Next we wish to say a few words about the relationship between q-pseudoconvexity and q-completeness. Suppose that M is a domain in $\tilde{M}$ and that M has C^2 boundary and $\tilde{M}$ is Stein. Then it can be shown that if M is q-pseudoconvex then M is q-complete (see Eastwood and Vigna Suria [1]). We only remark here that if $q = 0$ then the proof that q-pseudoconvexity implies q-completeness is similar to the proof of Proposition 2.5.15 and makes use of the elementary fact that we can find a Stein neighbourhood of $\bar{M}$ in $\tilde{M}$ which embeds in the unit Euclidean disc in some $\mathbb{C}^N$ (cf. Lemma 7.2.20). Conversely, it can be shown that q-completeness implies q-pseudoconvexity.

In Chapter 11 we shall show that q-completeness implies existence theorems for the $\bar{\partial}$-operator. To be precise, we shall show that if M is q-complete, E is a holomorphic vector bundle on E and $\phi \in C^{r,s}(M,E)$ is $\bar{\partial}$-closed then there exists $\psi \in C^{r,s-1}(M,E)$ such that $\bar{\partial}\psi = \phi$ provided that $s \geq q+1$, $r \geq 0$. In particular, if M is Stein we can always solve the generalised Cauchy-Riemann equations on M.

Although it is not our intention to say very much here about q-pseudoconvexity or q-completeness in case $q \neq 0$, we remark that q-pseudoconvexity may be regarded as a measure of how far away a domain is from being s.L.p. Moreover, the concept may be related to extension problems in complex analysis. See, for example, Eastwood and Vigna Suria [1] and Andreotti and Grauert [1]. We give one example to show how we may construct q-complete spaces, $q \neq 0$.

Example 4. (see also Griffiths [1; Theorem H], Serre [1], Simha [1], Vesentini [1]).

Let M be a p-complete manifold and $f_1,\dots,f_{q+1} \in A(M)$. Set $Z = Z(f_1,\dots,f_{q+1})$. Then we claim that $Y = M \setminus Z$ is (p+q)-complete. In particular if M is Stein and $q = 0$, Y is 0-complete and therefore Stein by the result cited above. Of course, in this case it is easy to verify directly that Y is holomorphically convex and therefore Stein as $f_1^{-1} \in A(Y)$.

Suppose ϕ is a strictly p-psh exhaustion function on M. Choose a C^∞ function $g\colon \mathbb{R} \to \mathbb{R}$ such that

1. $g(t), g'(t), g''(t) > 0$, $t \in \mathbb{R}$.

2. $g(t) \to \infty$ as $t \to \infty$.

3. $g\phi > \sum_{i=1}^{q+1} |f_i|^2$ on Y.

Set $\theta = g\phi - \log \sum_{i=1}^{q+1} |f_i|^2$. Since $L\left(\log \sum_{i=1}^{q+1} |f_i|^2\right)$ is positive semi-definite with at most q positive eigenvalues we see easily that θ is strictly (p+q)-psh on Y. It is sufficient to show that θ is an exhaustion function on Y. That is, $Y_c = \{x \in Y: \theta(x) < c\}$ is a relatively compact subset of Y for all $c \in \mathbb{R}$. Suppose $\theta(x) < c$. Certainly,

$$\sum_{i=1}^{q+1} |f_i|^2 \geq e^{g(\phi)-c} > e^{-c} > 0 .$$

Consequently, $Y_c \subset M \setminus U$, where U is an open neighbourhood of Z. On the other hand

$$e^{g(\phi)}/g(\phi) < e^c$$

and so $g(\phi) < e^c$ on Y_c. Hence $Y_c \subset M_C$, $C = g^{-1}(e^c)$. It follows that Y_c is a relatively compact subset of Y.

Finally, we conclude this section with a few brief remarks about the Bergman kernel function of a complex manifold. Given an m-dimensional complex manifold M, let

$$L^2(M) = \{f \in \Omega^m(M): \int_M f \wedge \bar{f} < \infty\}.$$

Then $L^2(M)$ has the structure of a Hilbert space with inner product defined by $(f,g) = \int_M f \wedge \bar{g}$. As in §6, Chapter 2 we can construct a Bergman kernel function $K(z,\bar{\zeta})$ for $L^2(M)$ and then $K(z,\bar{z})$ defines a smooth section of $C^{m,m}(M)$ (cf. Proposition 2.6.6). If instead, we start with an Hermitian metric on M and corresponding measure $d\lambda$ on M, we may define $L^2(M) = \{f \in A(M): \int |f|^2 d\lambda < \infty\}$. In this case we may show that there exists a Bergman kernel function $K(z,\bar{\zeta})$ for $L^2(M)$ and that for important examples Log $K(z,\bar{z})$ will be a strictly q-psh exhaustion function on M (cf. Proposition 2.6.8).

CHAPTER 6. SHEAF THEORY

Introduction

In Section 1 of this chapter we present the basic definitions and constructions of sheaf theory with many motivating examples. In section 2 we give an application of sheaf theory to prove the existence and uniqueness of the envelope of holomorphy of a Riemann domain. In section 3 we define the sheaf cohomology groups of a sheaf of groups over a paracompact space using fine resolutions. Amongst the most important results we prove are Leray's theorem and the existence of a canonical, natural isomorphism between Čech cohomology and sheaf cohomology. We conclude with a number of important examples and computations involving the 1st. Chern class.

§1. Sheaves and presheaves

Our aim in this section is to develop the theory of sheaves and show how it provides a unifying topological framework for the study of a diverse range of structures on topological spaces. Our presentation will be geared towards applications in complex analysis and the reader may consult Godement [1] or Tennison [1] for more extensive and general expositions of the theory of sheaves.

Let X be a topological space with topology of open sets $\mathcal{U}$.

Definition 6.1.1. A *presheaf of groups* on X is a collection of groups G(U), one for each $U \in \mathcal{U}$, together with group homomorphisms $r_{VU}: G(U) \to G(V)$, defined for $V, U \in \mathcal{U}$ and $V \subset U$, such that

1. If $U = \emptyset$, G(U) is the zero group.

2. For all $U \in \mathcal{U}$, r_{UU} is the identity.

3. For $U, V, W \in \mathcal{U}$ and $W \subset V \subset U$, we have $r_{WV}r_{VU} = r_{WU}$.

We usually denote the presheaf by $\{G(U), r_{VU}\}$ or just G.

Remarks.

1. Replacing the word "group" everywhere by "set", "ring", "field", "algebra", etc. we may define presheaves of sets, rings, fields, algebras, etc. We shall assume these definitions in the sequel.

2. The homomorphisms r_{VU} occuring in the definition are usually called *restriction homomorphisms*. In all our examples they will be restriction maps and we therefore generally omit any explicit specification.

As we shall soon see, many basic structures in analysis can be formulated in terms of presheaves.

Examples.

1. For each $U \in \mathcal{U}$, let $C(U)$ denote the ring of continuous $\mathbb{C}$-valued functions on U. Given $U,V \in \mathcal{U}$, with $V \subset U$, define $r_{VU}: C(U) \to C(V)$ to be restriction of continuous functions on U to V. Then $C_X = \{C(U), r_{VU}\}$ is a presheaf of rings on X: The *presheaf of continuous* $\mathbb{C}$*-valued functions* on X.

2. Let Γ be a ring with discrete topology. For each $U \in \mathcal{U}$, let $C(U,\Gamma)$ denote the ring of continuous Γ-vslued functions on U. Defining r_{VU} as restriction, $V \subset U$, the set $\Gamma_X = \{C(U,\Gamma), r_{VU}\}$ is a presheaf of rings on X: The *presheaf of locally constant* Γ*-valued functions* on X.

3. Suppose X has the structure of a differential manifold. Let $C^k(U)$ denote the ring of C^k $\mathbb{C}$-valued functions on U, $U \in \mathcal{U}$, $1 \le k < \infty$. Then $C^k_X = \{C^k(U), r_{VU}\}$ is a presheaf of rings on X: The *presheaf of* C^k $\mathbb{C}$*-valued functions* on X. We shall let $D_X = \{C^\infty(U), r_{VU}\}$ denote the *presheaf of* C^∞ $\mathbb{C}$*-valued functions* on X.

4. Suppose X has the structure of a complex manifold. We let $O_X = \{A(U), r_{VU}\}$ denote the presheaf of analytic $\mathbb{C}$-valued functions on X. In the sequel we usually refer to O_X as the *Oka presheaf of X*. For each U let S(U) denote the multiplicatively closed subset of A(U) consisting of all analytic functions on U which do not vanish identically on any component of U. Set $M(U) = A(U)_{S(U)}$. That is, M(U) is the quotient ring of A(U) with respect to the multiplicative system S(U) (See Zariski and Samuel [1; page 46] and observe that if U is connected M(U) is just the quotient field of A(U)). Then, defining r_{VU} as restriction, $M_X = \{M(U), r_{VU}\}$ is a presheaf of rings on X: The *presheaf of meromorphic functions* on X.

5. Let Z be an analytic subset of the complex manifold X. For each $U \in \mathcal{U}$, let $I_Z(U) = \{f \in A(U): f|Z \equiv 0\}$. Then $I_Z = \{I_Z(U), r_{VU}\}$ is a

presheaf of ideals of O_X. That is, for each $U \in \mathcal{U}$, $I_Z(U)$ is an ideal of $A(U)$. We call I_Z the *ideal presheaf* of Z.

6. Let E denote a holomorphic vector bundle over X. For each $U \in \mathcal{U}$, we let $\underline{E}(U)$ denote the space of holomorphic sections of E over U. Each $\underline{E}(U)$ is an $A(U)$-module in the obvious way and the presheaf $\underline{E} = \{\underline{E}(U), r_{VU}\}$ is thus an example of a *presheaf of O_X-modules*. We call $\underline{E}$ the *presheaf of holomorphic sections* of E. In case E is a smooth, not necessarily holomorphic, vector bundle over X, we let $\underline{E}_\infty = \{\underline{E}_\infty(U), r_{VU}\}$ denote the presheaf of C^∞ sections of E. Thus, $\underline{E}_\infty$ is an example of a presheaf of D_X-modules and, of course, the construction works for any differential manifold X.

Whilst a presheaf contains essentially all the information about a particular structure on a topological space, it is a large, seemingly cumbersome, object. We now describe the process of "sheafification" whereby out of every presheaf we can construct a topological space in such a way that, for all important examples, no information is lost.

Let $R = \{R(U), r_{VU}\}$ be a presheaf of rings on the topological space X. Fix $x \in X$. We define an equivalence relation $\sim_x$ on the rings $R(U)$ for which $x \in U$. Suppose that $U, V \in \mathcal{U}_x$ and $f \in R(U)$, $g \in R(V)$. We say f is equivalent to g at x, $f \sim_x g$, if and only if there exists $W \in \mathcal{U}_x$, $W \subset U \cap V$, such that

$$r_{WU}(f) = r_{WV}(g) .$$

Using conditions 2 and 3 of Definition 6.1.1, the reader may easily verify that $\sim_x$ is an equivalence relation. We denote the set of $\sim_x$ equivalence classes by R_x. The set R_x inherits the structure of a ring from the rings $R(U)$ and we let $r_{U,x}\colon R(U) \to R_x$ denote the corresponding "equivalence class" ring homomorphisms, defined for $U \in \mathcal{U}_x$. In the sequel we often write f_x for $r_{U,x}(f)$, $f \in R(U)$, and call f_x the *germ* of f at x.

Examples.

7. If we let O denote the Oka presheaf of the complex manifold X, then O_x is just the ring of germs of analytic functions at x (see also Chapter 3, §1; Chapter 4, §1).

8. If we let M denote the presheaf of meromorphic functions on the complex manifold X, then M_x is the field of germs of meromorphic functions at x (see Chapter 3, §4; Chapter 4, §1).

Set

$$R = \bigcup_{x \in X} R_x$$

and let $\pi: R \to X$ denote the projection defined by mapping points in R_x to x. We now topologise R. Given $f \in R(U)$, we have a section $\tilde{f}$ (relative to π) of R over U defined by $\tilde{f}(x) = f_x$, $x \in U$. For a base of open sets for the topology of R, we take the collection of sets $\tilde{f}(U) \subset R$, over all $U \in \mathcal{U}$ and $f \in R(U)$. The reader may easily verify that this defines the base for a topology on R. Clearly the local sections $\tilde{f}: U \to R$, $f \in R(U)$, are continuous in this topology.

Lemma 6.1.2. With the above notation we have

1. $\pi: R \to X$ is a local homeomorphism.

2. The induced topology on $R_x \subset R$ is discrete for all $x \in X$.

Proof. Property 2 is immediate from 1. For 1 we note that $\pi\tilde{f}$ is the identity map on U for all $f \in R(U)$ and so $\tilde{f}$ maps U homeomorphically onto $\tilde{f}(U)$ with inverse $\pi|\tilde{f}(U)$. □

We call the topological space R, together with the projection map $\pi: R \to X$, the *sheafification* of the presheaf R or the sheaf associated to the presheaf R. We denote the sheaf by the triple (R,π,X) or, more usually, by the symbol R. The ring $R_x = \pi^{-1}(x)$ is called the *stalk* of the sheaf at x.

Let us summarise our construction. Given a presheaf $R = \{R(U), r_{VU}\}$ of rings on X we have a ring R_x naturally defined at each point $x \in X$. The disjoint union R of the rings R_x has the natural structure of a topological space in such a way that the local sections $\tilde{f}$ associated to $f \in R(U)$ are continuous and the projection $\pi: R \to X$ is a local homeomorphism. Each stalk R_x has the structure of a ring with discrete topology. The triple (R,π,X) is called a *sheaf of rings* on X. Clearly our construction works equally well for presheaves of sets, groups, algebras, etc. to yield sheaves of sets, groups, algebras, etc. Shortly we shall give a general definition of a sheaf which does not depend, a priori, on the existence of a presheaf.

Examples.

9. Let C_X denote the sheaf of rings associated to the presheaf C_X of continuous $\mathbb{C}$-valued functions on X. We call C_X the *sheaf of germs of continuous $\mathbb{C}$-valued functions* on X. Often we drop the subscript X and just write C (this remark applies also to subsequent examples). The stalk C_x is the ring of germs of continuous $\mathbb{C}$-valued functions at x, $x \in X$. Suppose that $A: U \to C$ is a continuous section of C over the open subset U of X. We claim that there exists a unique $a \in C(U)$ such that $\tilde{a} = A$. In other words, continuous sections of the sheaf correspond to continuous $\mathbb{C}$-valued functions. First notice that A determines a function $a: U \to \mathbb{C}$ defined by $a(x) = A(x)(x)$ (evaluation of the germ $A(x)$ at x). We must prove that a is continuous. Let $x \in U$ and observe that by definition of the topology on C we may find an open neighbourhood V of x, $V \subset U$, and $s \in C(V)$ such that $\tilde{s} = A|V$. But now $s = a|V$ and so a is continuous at x. Since x was an arbitrary point in U it follows that a is continuous on U. Finally a is the unique $\mathbb{C}$-valued function on U satisfying $\tilde{a} = A$ since the germ of a at x determines the value of a at x, $x \in U$. Clearly what we have said above for the presheaf C_X and corresponding sheaf C_X works equally well for the presheaves C^k_X, D_X and O_X and so we obtain the sheaves of rings

C^k_X: Sheaf of germs of C^k $\mathbb{C}$-valued functions on the differential manifold X.

$\mathcal{D}_X$: Sheaf of germs of C^∞ $\mathbb{C}$-valued functions on the differential manifold X.

$\mathcal{O}_X$: Sheaf of germs of analytic functions on the complex manifold X.

The sheaf $\mathcal{O}_X$ is often referred to as the *Oka sheaf* of X.

We remark the important fact that a continuous local section of C^k_X (resp. $\mathcal{D}_X$, $\mathcal{O}_X$) over an open subset U corresponds to a unique C^k (resp. C^∞, analytic) function on U. The proof is the same as for C_X.

Finally, we observe that $\mathcal{O}_X$ is an open subset of $\mathcal{D}_X$ and that each stalk $\mathcal{O}_x$ is a subring of $\mathcal{D}_x$. We say that $\mathcal{O}_X$ is a *subsheaf* (of rings) of $\mathcal{D}_X$. Generally, if (R,π,X) and (S,η,X) are sheaves of rings on X, we say that R is a *subsheaf* (of rings) of S if R is an open subset of S, $\eta|R = \pi$ and for all $x \in X$, R_x is a subring of S_x. Thus all the sheaves constructed above are subsheaves of C_X. The reader may care to formulate the analogous concept of *subpresheaf*.

10. Let Γ denote a ring with discrete topology and Γ_X be the presheaf of locally constant Γ-valued function on X. Then the corresponding sheaf, which we shall also denote by Γ_X, is homeomorphic to $X \times \Gamma$. We say that Γ_X is a *constant sheaf* (that is, topologically a product). Sections of Γ_X are Γ-valued functions on X which are constant on connected components of X.

11. Let Z be an analytic subset of the complex manifold X. We let $\mathcal{I}_Z$ denote the sheaf associated to the ideal presheaf I_Z of Z. Dropping the subscript Z, we see that for each $x \in X$, $\mathcal{I}_x$ is an ideal in $\mathcal{O}_x$. For this reason we refer to $\mathcal{I}$ as a *sheaf of ideals* (of $\mathcal{O}$). Observe that for $x \notin Z$, $\mathcal{I}_x = \mathcal{O}_x$ whilst if $x \in Z$, $\mathcal{I}_x \subsetneq \mathcal{O}_x$.

12. If E is a holomorphic vector bundle over the complex manifold X, we let $\underline{E}$ denote the sheaf of germs of holomorphic sections of E associated to the presheaf of holomorphic sections of E. We see here that for each $x \in X$, $\underline{E}_x$ is an $\mathcal{O}_x$-module and we refer to $\underline{E}$ as a *sheaf of $\mathcal{O}$-modules*. For our particular example, we see that $\underline{E}_x$ is a free $\mathcal{O}_x$-module of rank equal to the fibre dimension of E.

13. Let $\mathcal{M}$ denote the *sheaf of germs of meromorphic functions* on X associated to the presheaf $M = \{M(U), r_{VU}\}$ of meromorphic functions on X. Since $\mathcal{M}$ is a field for all $x \in X$, $\mathcal{M}$ is an example of a sheaf of fields. A continuous section of $\mathcal{M}$ over X is called a meromorphic function on X - see Definition 4.4.4. However, a continuous section of $\mathcal{M}$ over an open subset U of X need not correspond to an element of M(U). This is a reflection of the fact that elements of M(U) are all quotients of analytic functions defined on U whilst meromorphic functions on U need not be representable globally as a quotient of analytic functions. The simplest example is found by taking $X = U = P^1(\mathbb{C})$ and m any non-constant meromorphic function on $P^1(\mathbb{C})$.

14. Our final example concerns the topology of the sheaves $\mathcal{D}_X$ and $\mathcal{O}_X$. We shall prove that the topology on $\mathcal{O}_X$ is Hausdorff whilst that on $\mathcal{D}_X$ is not. First we prove that $\mathcal{O}_X$ is Hausdorff. Let $A, B \in \mathcal{O}_X$, $A \neq B$. Set $x = \pi(A)$, $y = \pi(B)$. If $x \neq y$, choose disjoint open neighbourhoods U, V of x, y and $a \in A(U)$, $b \in A(V)$ such that $\tilde{a}(x) = A$, $\tilde{b}(y) = B$. Clearly $\tilde{a}(U)$, $\tilde{b}(V)$ are disjoint open neighbourhoods of A, B. If $x = y$, choose a connected open neighbourhood U of x and $a, b \in A(U)$ such that $\tilde{a}(x) = A$, $\tilde{b}(x) = B$. If $\tilde{a}(U) \cap \tilde{b}(U) \neq \emptyset$, uniqueness of analytic continuation implies that $a = b$ on U contradicting our assumption that $A \neq B$. Hence $\mathcal{O}_X$ is Hausdorff. To show that $\mathcal{D}_X$ is not Hausdorff it is enough to observe that for $n \geq 1$, we cannot separate the zero germ from the germ at zero, in $\mathbb{R}^n$, of the function γ defined by

$$\begin{aligned} \gamma(x_1,\ldots,x_n) &= 0, \; x_n \leq 0 \\ &= \exp(-1/x_n^2), \; x_n > 0 \, . \end{aligned}$$

Since $\mathcal{O}_X$ is Hausdorff it follows that $\mathcal{O}_X$ has the structure of a complex manifold spread over X (the complex structure on $\mathcal{O}_X$ is induced from that on X via the local homeomorphism π). We shall exploit this fact in our construction of the envelope of holomorphy in §2.

Next we shall give the general definition of a sheaf and show how a presheaf is naturally associated to every sheaf.

Suppose that we are given a topological space $\mathcal{F}$ and local homeomorphism $\pi: \mathcal{F} \to X$. We shall say that $(\mathcal{F}, \pi, X)$ is a *sheaf of rings* on X if

1. The stalks $\mathcal{F}_x$ $(= \pi^{-1}(x))$ have the structure of a ring for each $x \in X$.

2. The ring operations are continuous in the topology on $\mathcal{F}$.

Condition 2 needs further elaboration: Suppose that U is any open subset of X and s, t are continuous sections of $\mathcal{F}$ over U. Then we require that $s \pm t$, st are continuous sections of $\mathcal{F}$ over U, where we define addition, subtraction and multiplication of sections using the ring structure in the stalks. Equivalently, we may take the product $\mathcal{F} \times \mathcal{F}$ over $X \times X$ and restrict to the diagonal $\Delta \subset X \times X$. Addition, subtraction and multiplication then define maps of $\mathcal{F} \times \mathcal{F} | \Delta$ to $\mathcal{F}$ which should be continuous.

Again it is straightforward to define sheaves of sets, groups, algebras, etc. and we omit formal definitions. In case $\mathcal{R}$ is a sheaf of rings over X, we say that a sheaf $\mathcal{S}$ on X is a *sheaf of $\mathcal{R}$-modules* if each stalk $\mathcal{S}_x$ has the structure of an $\mathcal{R}_x$-module and the module operations are continuous in the sense described above.

Suppose that $(\mathcal{F},\pi,X)$ is a sheaf of rings. For each $U \in \mathcal{U}$, we let $\mathcal{F}(U)$ denote the space of continuous sections of $\mathcal{F}$ over U. Then $\mathcal{F}(U)$ is a ring and, defining restriction homomorphisms in the obvious way, we see that $\mathcal{F}' = \{\mathcal{F}(U), r_{VU}\}$ is a presheaf of rings on X.

Proposition 6.1.3. Let $(\mathcal{F},\pi,X)$ denote a sheaf of rings on X. Then the sheafification of the presheaf $\mathcal{F}' = \{\mathcal{F}(U), r_{VU}\}$ is equal to $\mathcal{F}$.

Proof. We leave this as an elementary exercise for the reader. □

Example 15. The presheaf $\mathcal{M}'$ associated to the sheaf $\mathcal{M}$ of germs of meromorphic functions on X is not generally equal to the presheaf M of meromorphic functions on X. However, it is clear that M and $\mathcal{M}'$ have the common sheafification $\mathcal{M}$.

Suppose that R is a presheaf of rings on X with associated sheaf $\mathcal{R}$. We have already seen that for all our examples, except that of meromorphic functions, $\mathcal{R}' = R$. Necessary and sufficient conditions for $\mathcal{R}'$ to equal R are given by the following elementary lemma the proof of which we omit.

Proposition 6.1.4. Let R be a presheaf of rings on X with associated sheaf $\mathcal{R}$. Then $\mathcal{R}' = R$ if and only if given any family $\{U_i : i \in I\} \subset \mathcal{U}$ and corresponding $s_i \in R(U_i)$ such that $s_i = s_j$ on U_{ij} for all $i,j \in I$, there exists a unique $s \in R(\cup U_i)$ such that $s|U_i = s_i$, $i \in I$.

Remarks on terminology and notation. In the literature a sheaf is often defined to be a presheaf which is equal to the presheafification of its associated sheaf. What we have called a sheaf is then referred to as the *espace étalé* of the sheaf (or presheaf). In the sequel we usually use the same notation for the sheaf and its associated presheaf. By virtue of Propositions 6.1.3, 6.1.4 this will not lead to confusion as, with the exception of the sheaf of germs of meromorphic functions, all our basic examples satisfy the conditions of Proposition 6.1.4. Aside from sheaves of sections of vector bundles and constant sheaves, we generally use script letters to denote sheaves and follow the notation developed earlier for our examples on sheaves.

For the remainder of this section we shall be considering morphisms of sheaves and various constructions involving sheaves. For the sake of brevity we restrict attention to sheaves of rings and modules noting that all our definitions generalise straightforwardly to sheaves of groups, fields, algebras, etc.

Morphisms of sheaves. Let $(\mathcal{F},\pi,X)$ and $(\mathcal{F}',\pi',X)$ be sheaves of rings on X. A *sheaf morphism* from $\mathcal{F}$ to $\mathcal{F}'$ is a continuous map $A: \mathcal{F} \to \mathcal{F}'$ covering the identity on X such that if $A_x: \mathcal{F}_x \to \mathcal{F}'_x$ denotes the map induced by A on the stalks at x then A_x is a ring homomorphism for all $x \in X$. We say that A is a *sheaf isomorphism* if A is a homeomorphism and A and A^{-1} are sheaf morphisms.

Remarks.

1. We shall often refer to a sheaf morphism between sheaves of rings (or groups, algebras, etc.) as a *sheaf homomorphism* or just *homomorphism*. In case $\mathcal{F}$, $\mathcal{F}'$ are $\mathcal{S}$-modules, where $\mathcal{S}$ is a sheaf of rings on X, we refer to a sheaf morphism between $\mathcal{F}$ and $\mathcal{F}'$ as an *$\mathcal{S}$-module homomorphism.*

2. Notice that a sheaf morphism is a local homeomorphism and therefore an open mapping.

We may similarly define morphisms between presheaves. Indeed, suppose $R = \{R(U), r_{VU}\}$ and $S = \{S(U), s_{VU}\}$ are presheaves of rings on X. Then a morphism $a: R \to S$ consists of a family $\{a_U: R(U) \to S(U): U \in \mathcal{U}\}$ of ring homomorphisms which are compatible with the restriction homomorphisms. That is, for $U,V \in \mathcal{U}$, $U \supset V$, we have the commutative diagram

$$\begin{array}{ccc} R(U) & \xrightarrow{a_U} & S(U) \\ \Big\downarrow r_{VU} & & \Big\downarrow s_{VU} \\ R(V) & \xrightarrow{a_V} & S(V) \end{array}$$

The reader may easily verify that a morphism a: R → S of presheaves induces a unique shesf morphism A: $\mathcal{R} \to \mathcal{S}$ between the sheaves associated to R and S. Conversely, any sheaf morphism A: $\mathcal{R} \to \mathcal{S}$ is associated to a unique presheaf morphism a: $\mathcal{R}' \to \mathcal{S}'$ between the presheaves associated to $\mathcal{R}$ and $\mathcal{S}$. We frequently use these observations in our construction of sheaf morphisms and indeed in the following examples all the sheaf morphisms are constructed first at the presheaf level.

Examples.

16. Let E and F be holomorphic vector bundles over the complex manifold X and A: E → F be a holomorphic vector bundle map. Then $\underline{E}$ and $\underline{F}$ are $\mathcal{O}_X$-modules and A induces in the obvious way an $\mathcal{O}_X$-module homomorphism $\underline{A}$: $\underline{E} \to \underline{F}$.

17. Let X be a differential manifold and for p ≥ 0 let $\underline{C}^p$ denote the sheaf of germs of C^∞ sections of the bundle $\wedge^p_{\mathbb{C}} \mathcal{T}^* X$ of complex p-forms. Note that $\underline{C}^0 = \mathcal{D}_X$ and each $\underline{C}^p$ has the structure of a $\mathcal{D}_X$-module. The constant sheaf $\mathbb{C}$ is a subsheaf of $\mathcal{D}_X$ and so the sheaves $\underline{C}^p$ have the structure of $\mathbb{C}$-modules. For p ≥ 0, exterior differentiation induces a morphism d: $\underline{C}^p \to \underline{C}^{p+1}$ of $\mathbb{C}$-modules. Observe that d is certainly not a morphism of $\mathcal{D}_X$-modules.

18. Let X be a complex manifold and for p,q ≥ 0 let $\underline{C}^{p,q}$ denote the sheaf of germs of C^∞ sections of the bundle $\wedge^{p,q}(X)'$ of complex (p,q)-forms. As in example 17, the operators ∂ and $\bar{\partial}$ induce $\mathbb{C}$-module homomorphisms

$$\partial: \underline{C}^{p,q} \to \underline{C}^{p+1,q}$$

$$\bar{\partial}: \underline{C}^{p,q} \to \underline{C}^{p,q+1} .$$

Since $\bar{\partial}\mathcal{O}_X = 0$, we see that $\bar{\partial}$: $\underline{C}^{p,q} \to \underline{C}^{p,q+1}$ is actually an $\mathcal{O}_X$-module homomorphism. Similarly ∂ is an $\bar{\mathcal{O}}_X$-module homomorphism where $\bar{\mathcal{O}}_X$ denotes the sheaf of germs of anti-holomorphic functions on X.

19. Let X be a complex manifold and for p ≥ 0 let Ω^p denote the sheaf of germs of holomorphic sections of $\wedge^{p,0}(M)'$. Since $\bar{\partial}(\Omega^p) = 0$,

we see that the operator ∂ induces a $\mathbb{C}$-module homomorphism

$$\partial : \Omega^p \to \Omega^{p+1}, \ p \geq 0.$$

Definition 6.1.5. A sequence $F \xrightarrow{A} G \xrightarrow{B} H$ of sheaves is said to be *exact* at G if the sequence $F_x \xrightarrow{A_x} G_x \xrightarrow{B_x} H_x$ of rings is exact for all $x \in X$.

Remark. We may define a sequence $R \xrightarrow{a} S \xrightarrow{b} T$ of presheaves to be exact at S if the sequence $R(U) \xrightarrow{a_U} S(U) \xrightarrow{b_U} T(U)$ of rings is exact for all $U \in \mathcal{U}$. It is most important to note that presheaf exactness is not equivalent to sheaf exactness and a sheaf exact sequence will *not* generally give rise to a presheaf exact sequence (the converse is always true). In fact sheaf exactness is very much a local matter whilst presheaf exactness is essentially global. In §3 we show how sheaf cohomology enables us to measure the deviation from presheaf exactness of a given sheaf exact sequence. We should also mention the important class of coherent $\mathcal{O}_X$-modules that we introduce in Chapter 7 for which it is true that sheaf exactness implies "local" presheaf exactness.

A sequence $0 \to F \xrightarrow{A} G \xrightarrow{B} H \to 0$ of sheaves is said to be *short exact* if each of the sequences $0 \to F_x \xrightarrow{A_x} G_x \xrightarrow{B_x} H_x \to 0$ is a short exact sequence, $x \in X$. We may similarly define exactness for general sequences and we follow the usual notational conventions.

Examples.

20. Let $0 \to E \xrightarrow{A} F \xrightarrow{B} G \to 0$ be a short exact sequence of holomorphic vector bundles over the complex manifold X. Then the corresponding sequence $0 \to \underline{E} \xrightarrow{A} \underline{F} \xrightarrow{B} \underline{G} \longrightarrow 0$ of sheaves is a short exact sequence of $\mathcal{O}_X$-modules.

21. Let X be an n-dimensional differential manifold. The *de Rham complex* is the sequence of $\mathbb{C}$-modules given by exterior differentiation:

$$0 \to \mathbb{C} \xrightarrow{i} \mathcal{D}_X \xrightarrow{d} \underline{C}^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{C}^{n-1} \xrightarrow{d} \underline{C}^n \to 0 .$$

(i denotes inclusion).

The Poincaré lemma implies that the de Rham complex is sheaf exact. However, the de Rham complex is generally not presheaf exact as we shall now show. Suppose that X is compact, oriented and without boundary. We prove that the de Rham complex is not presheaf exact at $\underline{C}^n$. For this it is enough to find an n-form on X (that is, continuous section of $\underline{C}^n$ over X) which is not the exterior derivative of an (n-1)-form on X. Choose any n-form ϕ on X such that $\int_X \phi \neq 0$ (Such forms always exist with support in a coordinate chart). Now $d\phi = 0$ and so if the de Rham complex is presheaf exact at $\underline{C}^n$ there exists an (n-1)-form ψ on X such that $d\psi = \phi$. But this cannot be since by Stokes' theorem $\int_X \phi = \int_X d\psi = \int_{\partial X} \psi = 0$. We shall see in §3 that the obstruction to ϕ being the boundary of an (n-1)-form is topological and lies in $H^n(X,\mathbb{C})$ - the nth. cohomology group of X. More precisely, ϕ determines an element $[\phi] \in H^n(X,\mathbb{C})$ and ϕ is a boundary if and only if $[\phi] = 0$.

22. Let X be an n-dimensional complex manifold. For p,q ≥ 0 we have the *Dolbeault complexes*

$$0 \to \Omega^p \xrightarrow{i} \underline{C}^{p,0} \xrightarrow{\bar\partial} C^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \underline{C}^{p,n-1} \xrightarrow{\bar\partial} \underline{C}^{p,n} \to 0$$

$$0 \to \bar\Omega^q \xrightarrow{i} \underline{C}^{0,q} \xrightarrow{\partial} \underline{C}^{1,q} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underline{C}^{n-1,q} \xrightarrow{\partial} \underline{C}^{n,q} \to 0 \quad .$$

Here i denotes inclusion and $\bar\Omega^q$ denotes the sheaf of germs of anti-holomorphic sections of the anti-holomorphic bundle $\Lambda^{0,q}(X)'$. In case p = 0, we obtain the important complex

$$0 \to \mathcal{O}_X \to \mathcal{D}_X \xrightarrow{\bar\partial} \underline{C}^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \underline{C}^{0,n-1} \xrightarrow{\bar\partial} \underline{C}^{0,n} \to 0$$

which relates the Oka sheaf to the sheaves $\underline{C}^{0,q}$.

It is an immediate consequence of the Dolbeault-Grothendieck lemma (Theorem 5.8.1) that the Dolbeault complexes are all exact.

23. It follows from example 19 that if X is an n-dimensional complex manifold we have a complex

$$0 \to \mathbb{C} \to \mathcal{O}_X \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^{n-1} \xrightarrow{\partial} \Omega^n \to 0$$

of $\mathbb{C}$-modules. We shall now prove that this sequence is exact.

Exactness at $\mathcal{O}_X$ is clear since $d|\mathcal{O}_X = \partial$. Now suppose $p > 0$ and let $\phi \in \Omega^p$ be ∂-closed. Since $\bar{\partial}\phi = 0$ and $d = \partial + \bar{\partial}$, $d\phi = 0$ and we see by the Poincaré lemma that there exists $\psi \in C^p$ such that $d\psi = \phi$. Since $d\psi = \partial\psi + \bar{\partial}\psi \in C^{p,0}$ we see that $\psi \in C^{p-1,0}$ and $\bar{\partial}\psi = 0$. That is $\psi \in \Omega^{p-1}$ and $\phi = \partial\psi$. Hence the sequence is exact.

For the next few paragraphs we consider some general constructions involving sheaves, presheaves and morphisms.

Definition 6.1.6. Let $A\colon F \to G$ be a sheaf homomorphism and $\{a_U\} = \{a_U\colon F(U) \to G(U)\}$ denote the corresponding morphism of presheaves. We define the *presheaf kernel* of A, *presheaf cokernel* of A and *presheaf image* of A to be the presheaves given by $U \longmapsto \mathrm{Ker}(a_U)$, $U \longmapsto \mathrm{Coker}(a_U)$ and $U \longmapsto \mathrm{Im}(a_U)$ respectively. We denote the associated sheaves by Ker(A), Coker(A) and Im(A) respectively and refer to them as the (sheaf) *kernel, cokernel* and *image* of A respectively.

Remarks.

1. It is easy to see that Ker(A) is always equal to the presheaf kernel of A but that, in general, Im(A) and Coker(A) are not equal to the image and cokernel presheaves of A.

2. For all $x \in X$, we have $\mathrm{Ker}(A)_x \approx \mathrm{Ker}(A_x)$, $\mathrm{Coker}(A)_x \approx \mathrm{Coker}(A_x)$ and $\mathrm{Im}(A)_x \approx \mathrm{Im}(A_x)$.

3. We say that A is *injective* if Ker(A) = 0; *surjective* if Coker(A) = 0. By remark 2 this is equivalent to injectivity or surjectivity at the stalk level.

4. A sheaf sequence $F \xrightarrow{A} G \xrightarrow{B} H$ is exact at G if and only if Im(A) = Ker(B) (Here we are regarding Im(A), Ker(B) as subsheaves of G).

5. Associated to a sheaf homomorphism $A\colon F \to G$ we have the exact sheaf sequence

$$0 \to \mathrm{Ker}(A) \xrightarrow{i} F \xrightarrow{A} G \xrightarrow{q} \mathrm{Coker}(A) \to 0 \quad .$$

(Here i and q are induced from the inclusion and quotient maps respectively).

Suppose i: $F \to G$ is the inclusion map of F as a subsheaf of G. We define the *quotient sheaf* G/F to be the sheaf associated to the presheaf $U \to G(U)/F(U)$. We have the corresponding short exact sequence

$$0 \to F \xrightarrow{i} G \xrightarrow{q} G/F \to 0 .$$

We remark that G/F is isomorphic to Coker(i) and that for all $x \in X$, $(G/F)_x \approx G_x/F_x$.

Examples.

24. Let $\mathcal{O}$, $\mathcal{M}$ respectively denote the Oka-sheaf and sheaf of germs of meromorphic functions on the complex manifold X. We have the short exact sequence of sheaves of Abelian groups

$$0 \to \mathcal{O} \to \mathcal{M} \xrightarrow{q} \mathcal{M}/\mathcal{O} \to 0 .$$

We see that local sections of $\mathcal{M}/\mathcal{O}$ are just the principal parts of meromorphic functions. That is, if $m, m' \in \mathcal{M}(U)$ then m, m' determine the same section of $\mathcal{M}/\mathcal{O}$ if and only if $m - m' \in \mathcal{O}(U)$ $(=A(U))$. We can now give a sheaf theoretic formulation of the Cousin I problem (see Definition 3.4.9): The *data* for the Cousin I problem on X is a continuous section P of $\mathcal{M}/\mathcal{O}$ over X. The *Cousin I problem* (for P) is then to find a continuous section m of $\mathcal{M}$ over X such that $q(m) = P$.

25. Let $\mathcal{O}^*$, $\mathcal{M}^*$ denote the multiplicative sheaves of groups of units of $\mathcal{O}$, $\mathcal{M}$ on the complex manifold X. Thus, $\mathcal{O}^*_x$, $\mathcal{M}^*_x$ will be the groups of invertible germs in $\mathcal{O}_x$, $\mathcal{M}_x$ respectively (see §4, Chapter 3). We let $\mathcal{D}$ denote the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$. Then $\mathcal{D}$ is a sheaf of Abelian groups called the *sheaf of germs of divisors* on X. We have the short exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0 .$$

Observe that a continuous section of $\mathcal{D}$ over X is a *Cartier divisor* (Definition 4.5.9). We may now give a sheaf theoretic formulation of the Cousin II problem (Definition 3.4.10): The *data* for the Cousin II problem is a continuous section d of $\mathcal{D}$ over X. The *Cousin II problem* (for d) is to find a continuous section m of $\mathcal{M}^*$ such that $q(m) = d$.

If we take X to be a Riemann surface we see that $\mathcal{D}_x \cong \mathbb{Z}$ for all $x \in X$. However, $\mathcal{D}$ is far from being the constant sheaf $\mathbb{Z}$ as continuous sections of $\mathcal{D}$ are always non-zero on a *discrete* subset of X. In fact the "zero set" of a continuous section of $\mathcal{D}$ is always an *open* subset of X!

26. Let Z be an analytic subset of the complex manifold X. We start by defining the Oka or structure sheaf of Z. Let U be an open subset of X and $f,g \in \mathcal{O}_X(U)$. We say f and g are Z-equivalent if $f = g$ on $Z \cap U$. That is, if $f - g \in I_Z(U)$ (see Example 5). The set of Z-equivalence classes associated to U is isomorphic to $\mathcal{O}_X(U)/I_Z(U)$ and $U \to \mathcal{O}_X(U)/I_Z(U)$ defines a presheaf of rings on X. We denote the associated sheaf by $\mathcal{O}_Z$ and observe that $\mathcal{O}_{Z,x} = 0$, $x \notin Z$. We call $\mathcal{O}_Z$ the *Oka* or *structure sheaf* of Z. Now $\mathcal{O}_Z$ is just the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Z$ and we have the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 .$$

Any element of $\mathcal{O}_X(U)/I_Z(U)$ determines a well defined continuous function on $Z \cap U$. Consequently, a continuous section of $\mathcal{O}_Z$ over an open set U of X determines a well-defined continuous function on $Z \cap U$. We call such functions *analytic functions* on $Z \cap U$. It is easy to see that a function $f: Z \cap U \to \mathbb{C}$ will be analytic if and only if for every $x \in Z \cap U$ there exists an open neighbourhood V of x in X and $g \in A(V)$ such that $g|Z \cap U \cap V = f|Z \cap V$.

Definition 6.1.7. Let $\mathcal{R}$ be a sheaf of rings on X and $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{R}$-modules.

1. The *direct sum* $\mathcal{F} \oplus \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ is the sheaf of $\mathcal{R}$-modules associated to the presheaf $U \to \mathcal{F}(U) \oplus \mathcal{G}(U)$.

2. The *tensor product* $\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{R}$ is the sheaf of $\mathcal{R}$-modules associated to the presheaf $U \to \mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U)$.

Remarks.

1. It is easily seen that $(\mathcal{F} \oplus \mathcal{G})_x \approx \mathcal{F}_x \oplus \mathcal{G}_x$ and $(\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})_x \approx \mathcal{F}_x \otimes_{\mathcal{R}_x} \mathcal{G}_x$, $x \in X$ and we assume these properties in the sequel.

2. We denote the p-fold direct sum of the sheaf of $\mathcal{R}$-modules $\mathcal{F}$ by $\mathcal{F}^p$.

Examples.

27. Let E, F be holomorphic vector bundles over the complex manifold X. Then $\underline{E \oplus F} = \underline{E} \oplus \underline{F}$ and $\underline{E \otimes F} = \underline{E} \otimes_{\mathcal{O}} \underline{F}$ (equality here means, of course, up to natural isomorphism). Similar relations hold for smooth or continuous vector bundles.

28. The sheaf of germs of $\mathbb{C}^p$-valued holomorphic functions on a complex manifold is isomorphic to $\mathcal{O}^p$ (we say the sheaf is free of rank p - see Definition 6.1.8 below)

29. Let E be a holomorphic vector bundle of dimension p over the complex manifold X. Then $\underline{E}$ is *locally isomorphic* to $\mathcal{O}^p$. That is, we may find an open neighbourhood U of each $x \in X$ such that $\underline{E}|U \cong \mathcal{O}^p_U$. Here the restriction of sheaves to an open set has the obvious interpretation. In particular, $\mathcal{O}_X|U = \mathcal{O}_U$.

Definition 6.1.8. Suppose F is a sheaf of R-modules on X. We say that F is a *locally free* sheaf of R-modules of rank p if F is locally isomorphic to R^p. We say that F is *free* of rank p if $F \cong R^p$.

The next proposition is valid, with the aame proof, for smooth or continuous vector bundles.

Proposition 6.1.9. Let X be a complex manifold. There is a bijective correspondence between isomorphism classes of locally free sheaves of $\mathcal{O}$-modules of finite rank over X and holomorphic vector bundles over X.

Proof. We have already indicated in Example 29 that the sheaf of holomorphic sections of a holomorphic vector bundle is a locally free sheaf of $\mathcal{O}$-modules of finite rank. Suppose now that F is a locally free sheaf of $\mathcal{O}$-modules on X of rank p. Thus we have an open cover $\{U_i\}$ of X and corresponding $\mathcal{O}_{U_i}$-isomorphisms $\phi_i: F|U_i \to \mathcal{O}^p_{U_i}$.

Define $\phi_{ij}: \mathcal{O}^p_{U_{ij}} \to \mathcal{O}^p_{U_{ij}}$ by $\phi_{ij} = \phi_i\phi_j^{-1}$. Now ϕ_{ij} is an isomorphism of the $\mathcal{O}_{U_{ij}}$-module $\mathcal{O}^p_{U_{ij}}$ and so is given by a $p \times p$-matrix with holomorphic entries defined on U_{ij}. That is, ϕ_{ij} determines a holomorphic map $\phi_{ij}: U_{ij} \to GL(p,\mathbb{C})$. Clearly the $\{\phi_{ij}\}$ are the transition functions for a holomorphic vector bundle E on X. We leave it to the reader to check that $\underline{E} = F$. □

Example 30. In this example we follow the notation and assumptions of Examples 18 and 22. Thus for $p,q \geq 0$ we have an $\mathcal{O}$-homomorphism $\bar{\partial}$: $\underline{C}^{p,q} \to \underline{C}^{p,q+1}$ of sheaves over the complex manifold X. If E is a holomorphic vector bundle over X, $\underline{E}$ is a sheaf of $\mathcal{O}$-modules and so, since $\bar{\partial}$ is an $\mathcal{O}$-homomorphism, we may form

$$\bar{\partial}(= \bar{\partial} \otimes I)\colon \underline{C}^{p,q} \otimes_{\mathcal{O}} \underline{E} \to \underline{C}^{p,q+1} \otimes_{\mathcal{O}} \underline{E} .$$

But $\underline{C}^{p,q} \otimes_{\mathcal{O}} \underline{E} = \underline{C}^{p,q} \otimes_{\mathcal{G}} \underline{E}_{\infty} = \underline{\Lambda^{p,q}(M,E)'_{\infty}}$. Hence, as in §7 of Chapter 5, we have extended the $\bar{\partial}$-operator to E-valued forms. In the sequel we set $\underline{\Lambda^{p,q}(M,E)'_{\infty}} = \underline{C}^{p,q}(E)$ and $\underline{\Lambda^{p,0}(M,E)'} = \Omega^p(\underline{E})$. Using the Dolbeault-Grothendieck lemma as in Example 22, we find that the Dolbeault complexes

$$0 \to \Omega^p(\underline{E}) \xrightarrow{i} \underline{C}^{p,0}(E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \underline{C}^{p,n-1}(E) \xrightarrow{\bar{\partial}} \underline{C}^{p,n}(E) \to 0$$

are exact for $p \geq 0$.

Next we see how sheaves transform under maps of the underlying topological spaces.

Suppose f: X → Y is a continuous map of topological spaces and (F,π, Y) is a sheaf of rings on Y. We shall define a sheaf $f^{-1}F$ of rings on X called the *inverse image sheaf* of F. To this end, the stalk of $f^{-1}F$ at $x \in X$ will be equal, as a ring, to $F_{f(x)}$. We let $f^{-1}F$ be the disjoint union of the rings $F_{f(x)}$ over $x \in X$. It remains to topologise $f^{-1}F$. For a basis of open sets for the topology on $f^{-1}F$ we take the set of all images s(U), where U is an open subset of X and s is a section of $f^{-1}F$ over U such that πs: U → Y is continuous.

Examples.

31. Let Z be a subset of X with induced topology and i: Z → X denote the inclusion map. If F is a sheaf on X we call $i^{-1}F$ the *restriction* of F to Z and denote it by $F|Z$ or F_Z.

32. Let F be a sheaf of R-modules on Y and f: X → Y be continuous. Then $f^{-1}F$ is a sheaf of $f^{-1}R$-modules on X. Suppose that X, Y are complex manifolds, f is holomorphic and E is a holomorphic vector bundle over Y.

It is natural to ask for the relation between $f^{-1}\underline{E}$ and the sheaf of holomorphic sections of the pull-back bundle f^*E. Now $f^{-1}\underline{E}$ is an $f^{-1}\mathcal{O}_Y$-module rather than an $\mathcal{O}_X$-module. However, $\mathcal{O}_X$ is naturally an $f^{-1}\mathcal{O}_Y$-module - indeed we have a sheaf homomorphism of $f^{-1}\mathcal{O}_Y$ into $\mathcal{O}_X$ defined in the obvious way by composition of elements of $f^{-1}\mathcal{O}_Y$ with f. Hence we may form the sheaf

$$f^*\underline{E} = f^{-1}\underline{E} \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X .$$

Now $f^*\underline{E}$ is a sheaf of $\mathcal{O}_X$-modules and the reader may verify that $f^*\underline{E} = \underline{f^*E}$. Generally, for any sheaf F of $\mathcal{O}_Y$-modules, we define $f^*F = f^{-1}F \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$. and f^*F will then be a sheaf of $\mathcal{O}_X$-modules on X.

Next we turn to the push-forward of sheaves. Suppose $f: X \to Y$ is a continuous map and (F,π,X) is a sheaf of rings on X. We define the *direct image sheaf* f_*F to be the sheaf of rings on Y associated to the presheaf $U \to F(f^{-1}(U))$.

Example 33. If $f: X \to Y$ is a holomorphic map of complex manifolds and F is a sheaf of $\mathcal{O}_X$-modules on X then f_*F is a sheaf of $f_*\mathcal{O}_X$-modules on Y. The reader may verify that we have a canonical sheaf morphism of $\mathcal{O}_Y$ into $f_*\mathcal{O}_X$ and so f_*F has the natural structure of an $\mathcal{O}_Y$-module. If E is a holomorphic vector bundle over X, $f_*\underline{E}$ will not generally be the sheaf of sections of a holomorphic vector bundle over Y. A simple example is found by taking the sheaf of sections $\underline{\mathbb{C}}$ ($= \mathcal{O}$) of the trivial bundle $\underline{\mathbb{C}}$ over $\mathbb{C}$ and the map $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$. It is easily verified that $f_*\underline{\mathbb{C}}$ is not locally free (see also the discussion below and Chapter 7).

Direct image sheaves play a most important role in complex analysis but are considerably more difficult to describe and analyse than inverse image sheaves. The full analysis of direct image sheaves requires the machinery of spectral sequences (see Godement [1] and also Griffiths and Harris [1; Chapter 3]). Here we shall only describe the stalks of direct image sheaves and then only in the case when the underlying spaces and map satisfy additional conditions.

Proposition 6.1.10. Let (F,π,X) be a sheaf of rings on X and $f: X \to Y$ be continuous. Suppose that X, Y are locally compact and that f is proper (that is, inverse images of compact sets are compact). Then f_*F_y is naturally isomorphic to $F(f^{-1}(y))$, all $y \in Y$. ($F(f^{-1}(y))$ denotes the spaces of continuous sections of the sheaf F, restricted to $f^{-1}(y)$, over $f^{-1}(y)$).

Our proof follows that given in Godement [1]. First we need some preliminary lemmas which are of interest in their own right.

Lemma 6.1.11. Let $\{M_i: i \in I\}$ be a locally finite cover by closed sets of the topological space X. Suppose that (F,π,X) is a sheaf of rings on X and that for each $i \in I$ we are given a continuous section s_i of F_{M_i}. Then, if the $\{s_i\}$ satisfy the compatibility condition $s_i = s_j$ on $M_i \cap M_j$, there exists $s \in F(X)$ such that $s|M_i = s_i$.

Proof. First we remark that the lemma is trivial if the $\{M_i\}$ form an *open* cover of X. Clearly our conditions on the $\{s_i\}$ imply that there exists a unique section s of F over X which restricts to s_i on M_i. We must show that s is continuous. Now, given $x \in X$, we may find an open neighbourhood U of x which meets only finitely many M_i, say $M_1,\ldots,M_p$. As the M_i are closed, we may assume that U is chosen so that $x \in M_1 \cap \ldots \cap M_p$. Shrinking U further if necessary we may also suppose that there exists $t \in F(U)$ such that $t(x) = s(x) = s_1(x) = .. = s_p(x)$. By definition of the sheaf topology on F, there exist open neighbourhoods U_j of x such that $t = s_j$ on U_j, $1 \le j \le p$. We may suppose $U = U_j$, $1 \le j \le p$. Hence s and t are equal in $U \cap (M_1 \cup \ldots \cup M_p) = U$ and so s is continuous at x. □

Lemma 6.1.12. Let S be a closed subset of the paracompact space X and suppose that F is a sheaf on X and $s \in F(S)$ ($= F_S(S)$). Then there exists an open neighbourhood V of S in X and $\tilde{s} \in F(V)$ such that $\tilde{s}|S = s$.

Proof. By definition of the topology on F, we may find an open neighbourhood U in X of every point $x \in S$ and $t \in F(U)$ such that $t|U \cap S = s$. Hence, by the paracompactness of x, we may find a locally finite open cover $\{U_i: i \in I\}$ of S and sections $s_i \in F(U_i)$ such that $s_i|U_i \cap S = s$ for all $i \in I$. Take a refinement $\{V_i\}$ of U_i such that $\bar{V}_i \subset U_i$ for all i and let W be the subset of X consisting of all points x in $\cup_i\bar{V}_i$ such that if $x \in \bar{V}_i \cap \bar{V}_j$ then $s_i(x) = s_j(x)$. By Lemma 6.1.11, applied to $F|W$, the $\{s_i\}$ define a continuous section $\tilde{s}$ of F over W. We claim that W contains an open neighbourhood of S. Let $x \in S$. There exists an open neighbourhood V of x which meets only finitely many of the $\bar{V}_i$'s, say $\bar{V}_1,\dots,\bar{V}_p$. Shrinking V we may suppose that $x \in \bar{V}_1,\dots,\bar{V}_p$. Now $s_1(x) = \dots = s_p(x)$ and so, shrinking V further if necessary, we may suppose that $s_1,\dots,s_p$ are equal on V. Now observe that $V \subset W$. □

Lemma 6.1.13. Let $f: X \to Y$ be a proper continuous map between locally compact spaces. Fix $y \in Y$ and let V be any open neighbourhood of $f^{-1}(y)$ in X. Then there exists an open neighbourhood U of y in Y such that $f^{-1}(U) \subset V$. In other words, a fundamental system of open neighbourhoods for $f^{-1}(y)$ is given by $\{f^{-1}(U)$: U an open neighbourhood of y

Proof. The intersection $\cap f^{-1}(\bar{U})$ over all relatively compact neighbourhoods U of y in Y is clearly equal to $f^{-1}(y)$. Since each $f^{-1}(\bar{U})$ is compact, it follows that for some relatively compact neighbourhood U of y we must have $f^{-1}(\bar{U}) \cap (X \setminus V) = \emptyset$. (For an alternative proof of this lemma, see the exercises at the end of the section). □

Proof of Proposition 6.1.10. We first remark that we have a natural homomorphism $\theta: f_*F_y \to F(f^{-1}(y))$ defined in the obvious way. We must show that F is bijective. Let $\gamma \in F(f^{-1}(y))$. By Lemma 6.1.12, there exists an open neighbourhood V of $f^{-1}(y)$ and $\tilde{\gamma} \in F(V)$ such that $\tilde{\gamma}|f^{-1}(y) = \gamma$. By Lemma 6.1.13, we may assume that $V = f^{-1}(U)$, for some open neighbourhood U of y. But now $\tilde{\gamma}$ lies in the presheaf generating f_*F and so determines an element of f_*F_y. Clearly this construction gives the required inverse to θ. □

We now briefly look at the use of sheaf formalism in defining structures on topological spaces. Suppose X is a topological space and F is a subsheaf of $\mathbb{C}$-algebras of C_X (regarded as a sheaf of $\mathbb{C}$-algebras). We may think of F as defining a structure on X: Structure of all functions of type F (that is, functions of type F would be given by continuous sections of F). All the structures so far considered - analytic, smooth, etc. - are locally defined and we shall now exploit this fact to give sheaf theoretic definitions of various structures on topological spaces.

Example 34. Let X be a topological space and F be a subsheaf of $\mathbb{C}$-algebras of C_X. Suppose that F is *locally isomorphic* to the sheaf of germs of C^∞ functions on $\mathbb{R}^n$. Then X may be given the unique structure of a differential manifold with $\mathcal{D}_X = F$. Before proving our assertion we need to explain what is meant by (local) isomorphism of sheaves defined over different topological spaces. For our example, we require that we can find an open neighbourhood U of each point $x \in X$ and a homeomorphism $\phi: U \to \phi(U) \subset \mathbb{R}^n$ such that the induced map $\phi^*: C_{\phi(U)} \to C_U$ restricts to an isomorphism of $\mathcal{D}_{\phi(U)}$ with F_U. We see that if F is locally isomorphic to $\mathcal{D}$, we may find an open cover $\{U_i\}$ of X and corresponding homeomorphism $\phi_i: U_i \to \phi(U_i) \subset \mathbb{R}^n$ such that ϕ_i induces an isomorphism of $\mathcal{D}_{\phi(U_i)}$ with F_{U_i} for all i. It is now a straightforward matter to verify that $\{(U_i,\phi_i)\}$ defines a differential atlas on X and that the associated structure sheaf $\mathcal{D}_X = F$. The same argument works if F is locally isomorphic to the Oka-sheaf of $\mathbb{C}^n$ and in this case we find that X has the structure of an n-dimensional complex manifold with Oka sheaf equal to F.

Motivated by the example above we may now give an intrinsic definition of a (reduced) analytic space which generalises our earlier definition of analytic set. First we define the *local models* for analytic spaces: The local models for analytic spaces will be the set of all pairs $(Z,\mathcal{O}_Z)$, where Z is an analytic subset of an open subset of some $\mathbb{C}^n$ and $\mathcal{O}_Z$ is the structure sheaf of Z (restricted to Z). A *(reduced) analytic space* will then be a pair $(X,\mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a subsheaf of $\mathbb{C}$-algebras of $\mathcal{O}_X$ which is locally isomorphic to a local model. That is, we may find an open neighbourhood V of any point $x \in X$, a local model $(Z,\mathcal{O}_Z)$ and a homeomorphism of V onto Z such that ϕ induces an isomorphism of $\mathcal{O}_V$ with $\mathcal{O}_Z$ ("isomorphism" in the sense described in Example 34).

Remark. Unfortunately fibre products of (reduced) analytic spaces cannot generally be constructed within the category of (reduced) analytic spaces. The problem lies with the fact that the structure sheaves we obtain when attempting these constructions may have nilpotents. Thus, the category of analytic spaces has to be enlarged so that it is closed under fibre products. For this, instead of considering subsheaves of C_X, we consider sheaves of local $\mathbb{C}$-algebras on X:

A sheaf F on X is said to be a sheaf of local $\mathbb{C}$-algebras on X if F is a sheaf of $\mathbb{C}$-algebras and each F_x has a unique maximal ideal m_x with $F_x/m_x \cong \mathbb{C}$ for all $x \in X$. A simple example of a sheaf of local $\mathbb{C}$-algebras which is not a subsheaf of C_X is given by taking X to be the origin of $\mathbb{C}$ and $F = O_0/(z^2)$ (O_0 denotes the Oka sheaf of $\mathbb{C}$ restricted to 0). Let us now describe the local models for (unreduced) analytic spaces. A local model will be a pair (X, O_X), where X is an analytic subset of an open subset U of $\mathbb{C}^n$ and $O_X = O_U/(f_1,\dots,f_k)|X$, where $f_1,\dots,f_k \in I_X(U)$ and $X = Z(f_1,\dots,f_k)$. An *analytic space* is then defined to be a pair (X, O_X), where O_X is a sheaf of local $\mathbb{C}$-algebras on X which is locally isomorphic to a local model. For further details and examples the reader may refer to the introductory article by Malgrange [1].

Exercises.

1. For each open subset U of the topological space X let B(U) denote the ring of continuous bounded $\mathbb{C}$-valued functions on U. Show that $B_X = \{B(U), r_{VU}\}$ is a presheaf of rings on X with sheaficication C_X. Hence deduce that the presheaf associated to the sheafification of B_X is not in general equal to B_X.

2. Let F, G be sheaves of rings on X and $Hom(F,G)$ be the sheaf associated to the presheaf $U \to Hom(F_U, G_U)$, where $Hom(F_U, G_U)$ is the ring of sheaf homomorphisms from F_U to G_U. Show that for all $x \in X$, we have a natural map $Hom(F,G)_x \to Hom(F_x, G_x)$ which is, in general, neither injective nor surjective.

3. Let X be a complex manifold with Oka sheaf O and F be a locally free sheaf of O-modules of finite rank on X. Define the dual F^* of F to be the sheaf $Hom_O(F, O)$. Prove that

A) F^* is the sheaf of germs of sections of the dual of the holomorphic bundle associated to F (up to isomorphism).

B) $F^{**} \approx F$.

C) $Hom_O(F,G) \approx F^* \otimes G$, for any sheaf of O-modules G.

D) $Hom_O(F,G)_x \approx Hom(F_x, G_x)$, $x \in X$.

Show that similar results hold for locally free sheaves of C_X- or $\mathcal{D}_X$-modules.

4. Let Z be a closed subset of the topological space X and $i: Z \to X$ denote the inclusion map. If F is a sheaf on Z, set $\tilde{F} = i_*F$. We call $\tilde{F}$ the *trivial extension* of F to X or the sheaf on X obtained by extending F by zero outside Z. Show that

A) $\tilde{F}_x = 0,\ x \notin Z$

$\quad = F_x,\ x \in Z.$

B) $i^{-1}\tilde{F} \approx F.$

5. Let (F,π,X) be a sheaf of rings and $s \in F(X)$. Define the *support of s*, supp(s), to be $\{x \in X: s(x) \neq 0\}$. Show that supp(s) is a closed subset of X.

6. Let (F,π,X) be a sheaf of rings. Define the *support of F*, supp(F), to be $\{x \in X, F_x \neq 0\}$. Show that supp(F) need be neither open nor closed.

7. Let F be a locally free sheaf of $\mathcal{O}_Y$-modules on the complex manifold Y and suppose that $f: X \to Y$ is a holomorphic map of complex manifolds. Prove that

$$f_*(G \otimes_{\mathcal{O}} f^*F) \approx f_*G \otimes_{\mathcal{O}} F, \text{ for any sheaf } G \text{ of } \mathcal{O}_X\text{-modules on } X.$$

8. Let X,Y be complex manifolds and F, G be sheaves of $\mathcal{O}_X$-, $\mathcal{O}_Y$-modules, prespectively. Show that for any holomorphic map $f: X \to Y$, $\mathrm{Hom}(f^*G,F) \approx \mathrm{Hom}(G,f_*F)$ (We say that f^* and f_* are *adjoint functors* between the categories of $\mathcal{O}_X$- and $\mathcal{O}_Y$-modules). Deduce that we have canonical homomorphisms $G \to f_*f^*G$ and $f^*f_*F \to F$.

9. Let X be a complex manifold and set $K = \text{Kernel } \partial\bar{\partial}: \underline{C}^{0,0} \to \underline{C}^{1,1}$. Show that the complex

$$0 \to K \to \underline{C}^{0,0} \xrightarrow{\partial\bar{\partial}} \underline{C}^{1,1} \xrightarrow{d} \underline{C}^{1,2} \oplus \underline{C}^{2,1} \xrightarrow{d} \cdots$$

is sheaf exact (Hint: For exactness at $\underline{C}^{1,1}$ the argument used in Example 23 may be helpful).

10. Let $f: X \to Y$ be a continuous map of topological spaces. Prove

a) If $a: F \to G$ is a morphism of sheaves over Y then we have a canonical sheaf map $f^{-1}(a): f^{-1}F \to f^{-1}G$ of sheaves over X.

b) f^{-1} is *exact*. That is, given a short exact sequence

$0 \to F \xrightarrow{a} G \xrightarrow{b} H \to 0$ of sheaves over Y, the sequence

$0 \to f^{-1}F \xrightarrow{f^{-1}(a)} f^{-1}G \xrightarrow{f^{-1}(b)} f^{-1}H \to 0$ is exact.

11. Same assumptions as Q10. Show that if $a: F \to G$ is a morphism of sheaves over X then we

a) have a canonical sheaf map $f_*(a): f_*F \to f_*G$ of sheaves over Y.

b) f_* is *left exact*. That is, given an exact sequence

$0 \to F \xrightarrow{a} G \xrightarrow{b} H$ of sheaves over X, the sequence

$0 \to f_*F \xrightarrow{f_*(a)} f_*G \xrightarrow{f_*(b)} f_*H$ is exact.

c) In general f_* is not exact.

(Hint for c): Take $X = \mathbb{C}^2 \setminus \{0\}$, $Y = \mathbb{C}$ and define $f(x,y) = x$. Let I denote the ideal sheaf of $f^{-1}(0)$. Show that the f_* image of the sequence $O_X \to O_X/I \to 0$ is not exact).

12. Suppose that $f: X \to Y$ is a holomorphic map of complex manifolds. Prove that

a) If $a: F \to G$ is a morphism of sheaves of O_Y-modules over Y then we have a canonical sheaf morphism $f^*a: f^*F \to f^*G$ of sheaves of O_X-modules.

b) f^* is right exact.

c) f^* is generally not left exact.

13. Let X, Y be metrizable topological spaces and $f: X \to Y$ be a proper continuous map. Show that f is closed (that is, f-images of closed sets are closed). Deduce Lemma 6.1.13 in case X, Y are metrizable (as will always be the case if X, Y are manifolds).

14. (Koszul complex). Let E be a holomorphic vector bundle of rank q on the complex manifold M and suppose $s \in \Omega(E)$. Show that we have complexes of O-modules

$$0 \to O \xrightarrow{\beta_0} \Lambda^1\underline{E} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{q-1}} \Lambda^q\underline{E} \to 0$$

$$0 \leftarrow O \xleftarrow{\alpha_0} \Lambda^1\underline{E}^* \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_{q-1}} \Lambda^q\underline{E}^* \leftarrow 0$$

where $\beta_j(t) = s \wedge t$; $\alpha_j(t) = C_s t$, $0 \le j \le q-1$, and that these complexes are exact if (and only if) s is nowhere zero (see exercise 6, §1, Chapter 5).

§2. Envelope of holomorphy

In this section we wish to consider the following problem: Given a domain Ω in $\mathbb{C}^n$ can we find a "maximal" n-dimensional complex manifold $\hat{\Omega}$ containing Ω such that every analytic function on Ω has a unique extension to $\hat{\Omega}$? Example 6 of §4, Chapter 2 shows that we cannot require $\hat{\Omega}$ to be a domain in $\mathbb{C}^n$. It turns out that to solve our problem it is best to work within the category of domains spread over $\mathbb{C}^n$. It then becomes possible to give a particularly elegant solution using the formalism of sheaf theory. First we recall some definitions.

Definition 6.2.1. A manifold *spread* over $\mathbb{C}^n$ is a pair (Ω,π), where Ω is a connected, separable, Hausdorff space and $\pi: \Omega \to \mathbb{C}^n$ is a local homeomorphism (not necessarily surjective). We call π the *spreading* of Ω in $\mathbb{C}^n$.

Given a spread manifold (Ω,π), the map π induces a complex structure on Ω. In the sequel we always assume that Ω comes with this complex structure. We also often refer to (Ω,π) as "the manifold spread over $\mathbb{C}^n$", omitting reference to π. We remark that given a complex manifold Ω there may exist many different spreadings of Ω in $\mathbb{C}^n$ all of which induce the given complex structure on Ω.

Suppose (Ω,π), (Ω',π') are manifolds spread over $\mathbb{C}^n$ and $\theta: \Omega \to \Omega'$ is continuous. We say that θ is a *morphism* of spread manifolds if $\pi'\theta = \pi$. That is, if $\theta(\Omega_x) = \Omega'_x$ for all $x \in \pi(\Omega)$. Here we have set $\Omega_x = \pi^{-1}(x)$, $\Omega'_x = \pi'^{-1}(x)$. The reader may easily verify that a morphism of spread manifolds is *analytic* with respect to the induced complex structures and is also an *open mapping*.

Definition 6.2.2. Let Ω, Ω' be manifolds spread over $\mathbb{C}^n$ and $\theta: \Omega \to \Omega'$ be a morphism. We say that the pair (Ω',θ) is an *analytic extension* of Ω if every $f \in A(\Omega)$ extends uniquely across θ to an element $\tilde{f} \in A(\Omega')$. That is, there exists $\tilde{f} \in A(\Omega')$ such that $f = \tilde{f}\theta$.

Remark. We sometimes say that Ω' is an analytic extension of Ω if there exists a morphism θ such that (Ω',θ) is an analytic extension of Ω in the sense of Definition 6.2.2.

Definition 6.2.3. A manifold Ω spread over $\mathbb{C}^n$ is called a *domain of holomorphy* if for every analytic extension (Ω',θ) of Ω, θ is an isomorphism.

Remark. The reader may easily verify that a domain Ω in $\mathbb{C}^n$ is a domain of holomorphy according to Definition 6.2.3 if and only if it is according to Definition 2.4.1.

We may now state the main result of this section.

Theorem 6.2.4. Let (Ω,π) be a manifold spread over $\mathbb{C}^n$. Then there exists an analytic extension $(\hat{\Omega},\theta)$ of Ω such that

1. $\hat{\Omega}$ is a maximal analytic extension of Ω in the sense that if $(\tilde{\Omega},\gamma)$ is any other analytic extension of Ω, there exists a morphism $\eta: \tilde{\Omega} \to \hat{\Omega}$ making $(\hat{\Omega},\eta)$ an analytic extension of Ω.

2. $\hat{\Omega}$ is independent of the spreading π of Ω in $\mathbb{C}^n$. That is, if π_1, π_2: $\Omega \to \mathbb{C}^n$ are two spreadings of Ω in $\mathbb{C}^n$, compatible with the given complex structure on Ω, then the corresponding maximal analytic extensions $\hat{\Omega}_1$, $\hat{\Omega}_2$ given by 1 are isomorphic.

Definition 6.2.5. We call for any analytic extension $(\hat{\Omega},\theta)$ of Ω satisfying the conditions of Theorem 6.2.4 the *envelope of holomorphy of* Ω.

Proof of Theorem 6.2.4. Our proof follows that in Malgrange [2].

Step 1. Existence of $\hat{\Omega}$.

Let $\{f_i\}$ denote the set of elements in $A(\Omega)$ indexed by the set I. We let $\mathbb{C}^I$ denote the vector space of all functions from I to $\mathbb{C}$; addition and scalar multiplication defined coordinatewise. Let U be an open subset of $\mathbb{C}^n$. We say that a map $h = (h_i): U \to \mathbb{C}^I$ is analytic if each component function $h_i: U \to \mathbb{C}$ is analytic. As in §1 we may construct the sheaf $(\mathcal{O}^I,p,\mathbb{C}^n)$ of germs of analytic functions from $\mathbb{C}^n$ to $\mathbb{C}^I$. Exactly as in Example 14 of §1, $(\mathcal{O}^I,p)$ is a spread manifold over $\mathbb{C}^n$ (of course,

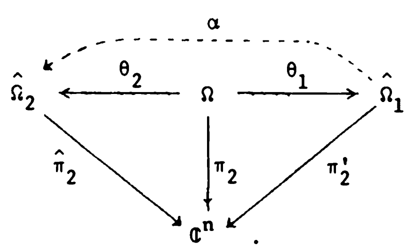
$\mathcal{O}^I$ will not be connected). We shall show that the envelope of holomorphy of Ω may be represented as a connected component of $\mathcal{O}^I$. First we define a morphism $\theta: \Omega \to \mathbb{C}^I$. Given $x \in \Omega$, choose an open neighbourhood U of x in Ω such that π restricts to a homeomorphism of U on $\pi(U) \subset \mathbb{C}^n$. Set $\phi = (\pi|U)^{-1}: \pi(U) \to \Omega$. For $i \in I$, set $\tilde{f}_i = f_i\phi \in A(\pi(U))$. Define $\theta(x) = (\tilde{f}_{i,x}) \in \mathcal{O}^I$, where $\tilde{f}_{i,x}$ denotes the germ of $\tilde{f}_i$ at x. It is clear from the definition of the topology on $\mathcal{O}^I$ that θ is continuous and so defines a morphism of spread manifolds. We let $\hat{\Omega}$ denote the connected component of $\mathcal{O}^I$ containing $\theta(\Omega)$ and set $\hat{p} = p|\hat{\Omega}$. Thus $(\hat{\Omega},\hat{p})$ is a manifold spread over $\mathbb{C}^n$.

Step 2. $(\hat{\Omega},\theta)$ is an analytic extension of Ω. We show that each $f_i \in A(\Omega)$ extends uniquely to an element $\hat{f}_i \in A(\hat{\Omega})$. First observe that for each $i \in I$ we have a projection map $\pi_i: \mathbb{C}^I \to \mathbb{C}$ which induces a morphism $\eta_i: \mathcal{O}^I \to \mathcal{O}$ of spread manifolds. Thus η_i maps the germ of a map of $\mathbb{C}^n$ into $\mathbb{C}^I$ to its ith. component. Let $z \in \hat{\Omega}$ and choose an open neighbourhood U of z in $\hat{\Omega}$ such that $\hat{p}|U$ is a homeomorphism. Set $\gamma = (\hat{p}|U)^{-1}$. Certainly $\pi_i\gamma$ is a section of $\mathcal{O}$ over $\hat{p}(U)$ and we let γ_i denote the corresponding analytic function on $\hat{p}(U)$. We define $\hat{f}_i|U = \gamma_i\hat{p}$. This construction clearly defines an element $\hat{f}_i \in A(\hat{\Omega})$ for each $i \in I$. Next we must prove that $\hat{f}_i\theta = f_i$ and that $\hat{f}_i$ is the unique element of $A(\hat{\Omega})$ satisfying this relation. Uniqueness is obvious by uniqueness of analytic continuation and the openness of $\theta(\Omega)$ in $\hat{\Omega}$. For the extension property suppose $x \in \Omega$ and choose an open neighbourhood U of x such that $\pi|U$ is a homeomorphism. Now $\theta(x) = (\tilde{f}_{i,x})$ and $\pi_i((\tilde{f}_{i,x})) = \tilde{f}_{i,x}$. Hence $\hat{f}_i(\theta(x)) = ((f.\pi^{-1})|\pi(U))\hat{p}\theta(x) = (f_i.\pi^{-1}|\pi(U))\pi(x) = f_i(x)$.

Step 3. $\hat{\Omega}$ is a maximal analytic extension of Ω. Suppose that $(\tilde{\Omega},\eta)$ is any analytic extension of Ω. We may repeat the construction of Step 1 to obtain an analytic extension $\hat{\tilde{\Omega}}$ of $\tilde{\Omega}$, with $\hat{\tilde{\Omega}}$ a connected component of $\mathcal{O}^I$. Observe that the construction used implies that $\hat{\tilde{\Omega}}$ necessarily contains the image of Ω by $\tilde{\theta}$ in $\mathcal{O}^I$. Since $\hat{\Omega}$ is defined as the connected component $\theta(\Omega)$ in $\mathcal{O}^I$ it follows immediately that $\hat{\Omega} = \hat{\tilde{\Omega}}$. Hence $\hat{\Omega}$ is a maximal analytic extension of Ω.

Step 4. $\hat{\Omega}$ is independent, up to isomorphism, of π. We first note that if $\tilde{\Omega}$ is an analytic extension of Ω then every analytic map $F: \Omega \to \mathbb{C}^m$, $m \geq 1$, extends uniquely to $\tilde{\Omega}$. This follows by applying the definition componentwise to F.

Let us suppose that we are given two spreadings π_1, π_2 of Ω with corresponding maximal analytic extensions $(\hat{\Omega}_1,\theta_1)$, $(\hat{\Omega}_2,\theta_2)$. By Step 3 and the above remark, π_2 extends to $\pi_2': \hat{\Omega}_1 \to \mathbb{C}^n$. We claim that π_2' spreads $\hat{\Omega}_1$ in $\mathbb{C}^n$. For this we first note that the tangent bundle $T\hat{\Omega}_1$ is analytically trivial since $\hat{\Omega}_1$ is spread in $\mathbb{C}^n$ by π_1. But if $T\hat{\Omega}_1 \cong \hat{\Omega}_1 \times \mathbb{C}^n$, it follows that we can take the derivative of π_2' to obtain a map $D\pi_2': \hat{\Omega}_1 \to L(\mathbb{C}^n,\mathbb{C}^n)$. Since π_2 is a spreading it follows that $q(x) = D\pi_2(\theta_1(x))$ is an isomorphism for all $x \in \Omega$. Hence we may define an analytic map $p: \Omega \to L(\mathbb{C}^n,\mathbb{C}^n)$ by $p(x) = q(x)^{-1}$, $x \in \Omega$. Applying our remark again we see that p extends to $\hat{p}: \hat{\Omega}_1 \to L(\mathbb{C}^n,\mathbb{C}^n)$. Form the composite maps $p.D\pi_2'$, $D\pi_2'.p$ (composition in $L(\mathbb{C}^n,\mathbb{C}^n)$). We see that both compositions are analytic and equal to the identity on $\theta_1(\Omega)$. Hence, by uniqueness of analytic continuation, they are equal to the identity on the whole of $\hat{\Omega}_1$. It follows that $D\pi_2'$ is invertible on $\hat{\Omega}_1$ and so, by the inverse function theorem, π_2' defines a spreading of $\hat{\Omega}_1$ in $\mathbb{C}^n$, compatible with the given complex structure on $\hat{\Omega}_1$.



Now $(\hat{\Omega}_1,\theta_1)$ is an analytic extension of (Ω,π_2), where we choose the spreading of $\hat{\Omega}_1$ given by π_2'. By the maximality property of $(\hat{\Omega}_2,\theta_2)$ it follows that there exists a morphism $\alpha: \hat{\Omega}_1 \to \hat{\Omega}_2$ satisfying $\alpha\theta_1 = \theta_2$. The map α is unique by uniqueness of analytic continuation. We may now repeat the above constructions to obtain a morphism $\beta: (\hat{\Omega}_2,\pi_1') \to (\hat{\Omega}_1,\hat{\pi}_1)$. It remains to prove that α and β are inverses of one another. This follows, again by uniqueness of analytic continuation, once we have noticed that

$$\beta\alpha(\theta_1(x)) = \theta_1(x); \quad \alpha\beta(\theta_2(x)) = \theta_2(x), \ x \in \Omega. \qquad \square$$

Corollary 6.2.5. Let $\tilde{\Omega}$ be an analytic extension of Ω. If $\tilde{\Omega}$ is a domain of holomorphy, $\tilde{\Omega}$ is the envelope of holomorphy of Ω.

Proof. Let $\hat{\tilde{\Omega}}$ denote the envelope of holomorphy of $\tilde{\Omega}$ given by Theorem 6.2.4. If $\tilde{\Omega}$ is a domain of holomorphy then $\hat{\tilde{\Omega}}$ is isomorphic to $\tilde{\Omega}$. But $\hat{\tilde{\Omega}}$ is a maximal analytic extension of $\tilde{\Omega}$ and hence of Ω. It follows that $\tilde{\Omega}$ is the envelope of holomorphy of Ω. □

Remarks.

1. The map $\theta: \Omega \to \hat{\Omega}$ constructed in Step 1 of the proof of Theorem 6.2.4 will not be injective unless $A(\Omega)$ separates points in Ω. This requirement is fulfilled of course if Ω is a domain in $\mathbb{C}^n$.

2. It should be noticed that the proof of Theorem 6.2.4 works for any subset of $A(\Omega)$. In particular, it gives us the maximum domain of continuation of any analytic function defined on Ω, representing it as a manifold spread over $\mathbb{C}^n$.

3. It should be appreciated that Theorem 6.2.4 is very much an existence and uniqueness theorem that gives no information about how to construct and represent envelopes of holomorphy in practice. Of course, our existence proof is very formal and elemenatary and not too much should be expected. In this regard it should be compared with the classical continuation proof such as is given in Hörmander [1; Theorem 5.4.5].

4. The envelope of holomorphy of a manifold spread over $\mathbb{C}^n$ may also be represented as the spectrum of the algebra $A(\Omega)$. That is, the envelope of holomorphy is isomorphic to the space of non-zero continuous homomorphisms of $A(\Omega)$ into $\mathbb{C}$. For this approach to the construction of the envelope of holomorphy we refer to Gunning and Rossi [1].

5. Finally we remark the fundamental result that a manifold spread over $\mathbb{C}^n$ is a domain of holomorphy if and only if it is holomorphically convex and if and only if it is isomorphic to its envelope of holomorphy. In particular, every domain of holomorphy is a Stein manifold. Proofs of these results, which use pseudo-convexivity methods, may be found in Gunning and Rossi [1] and Hörmander [1].

Exercises.

1. Let (Ω,π) be a manifold spread over $\mathbb{C}^n$. Show that if we let $\tilde{\Omega}$ denote the quotient space of Ω defined by the relation "x equivalent to y iff x and y cannot be separated by elements of $A(\Omega)$", then

A) $\tilde{\Omega}$ is naturally a manifold spread over $\mathbb{C}^n$.

B) $A(\tilde{\Omega})$ separates points in $\tilde{\Omega}$.

C) The envelopes of holomorphy of Ω and $\tilde{\Omega}$ are isomorphic.

2. Prove that a holomorphically convex manifold spread over $\mathbb{C}^n$ is a domain of holomorphy (Hint: Follow the proof given in Chapter 2).

3. Let (Ω,π) be a manifold spread over $\mathbb{C}^n$ and suppose $\Omega \neq \mathbb{C}^n$. Given a compact subset K of Ω, define $d(K) = \inf\{|z-\pi(\zeta)|: z \in \partial(\pi\Omega), \zeta \in K\}$ (see also §4, Chapter 2). Prove that if (Ω,π) is a domsin of holomorphy then $d(K) = d(\hat{K})$ for all compact subsets of Ω. Show also that if (Ω,π) is finitely sheeted and $d(K) = d(\hat{K})$ for all compact subsets K of Ω, then (Ω,π) is holomorphically convex and so a domain of holomorphy (the non-finitely sheeted case is more difficult and is treated in Gunning and Rossi [1]).

§3. Sheaf cohomology.

In §1 we showed how sheaf formalism provided a unifying topological framework for the description of a wide range of structures on topological spaces. In this section we introduce a powerful computational machine for the analysis of sheaves: *Sheaf Cohomology*. In essence our methods allow us to apply the highly systematised and powerful methods of homological algebra to problems in global complex analysis and algebraic geometry. The ideas we describe were introduced into complex analysis by H. Cartan (see H. Cartan [1,2]) and into algebraic geometry by J.P. Serre (see J.P. Serre [2]). This use of sheaves and sheaf cohomology has undoubtedly revolutionised and clarified both fields.

Our approach to sheaf cohomology will be to first develop a rather abstract, non-computable, theory which is valid for sheaves defined over paracompact spaces. We then relate this theory to the computable Čech theory by means of Leray's theorem.

Throughout this section we shall assume that all topological spaces are paracompact and, as always, Hausdorff. "Sheaf" will always refer to a sheaf of abelian groups unless the contrary is indicated.

Suppose that $0 \to F \xrightarrow{a} G \xrightarrow{b} H \to 0$ is a short exsct sequence of sheaves over X. Given an open set $U \subset X$, it is easily seen that the sequence

$$0 \to F(U) \xrightarrow{a_U} G(U) \xrightarrow{b_U} H(U)$$

is exact but that, in general, $b_U: G(U) \to H(U)$ will not be surjective (The section functor is left- but not right-exact). Just by reference to the Cousin I and II problems (Examples 24, 25, §1) we see the importance of finding a measure of how far the map $b_U: G(U) \to H(U)$ fails to be surjective. A satisfactory solution to this problem is the primary aim of sheaf cohomology theory. Our first task will be to describe a class of sheaves for which it is true that short exact sequences of sheaves transform into short exact sequences of groups under the section functor. That is, we shall be describing a class of sheaves for which sheaf and presheaf exactness are equivalent.

Definition 6.3.1. A sheaf (F,π,X) is said to be *soft* if for all closed subsets K of X and sections $s \in F(K)$, s extends to s continuous section of F over X. That is, the natural map $F(X) \to F(K)$ is surjective for all closed subsets K of X.

Proposition 6.3.2. Let $0 \to F \xrightarrow{a} G \xrightarrow{b} H \to 0$ be a short exact sequence of sheaves over X. Provided F is soft, the corresponding sequence of sections

$$0 \to F(X) \xrightarrow{a} G(X) \xrightarrow{b} H(X) \to 0$$

is exact.

Proof. First we remark that we shall now generally use the same notation for the map induced on sections and the corresponding sheaf morphism. If it is necessary to distinguish them, we use a "star" superscript for the map induced on sections.

We must show that for every $s \in H(X)$, there exists $t \in G(X)$ such that $b(t) = s$. Since $b\colon G \to H$ is surjective, we may find an open neighbourhood U_x of every $x \in X$ and $t_x \in G(U_x)$ such that $b(t_x) = s|U_x$. Hence, by the paracompactness of X, we may find a locally finite open cover $\{U_i : i \in I\}$ of X and family $\{t_i \in G(U_i)\}$ such that $b(t_i) = s|U_i$, $i \in I$. Choose an open refinement $\{V_i\}$ of $\{U_i\}$ such that $\bar{V}_i \subset U_i$, $i \in I$. Consider the set Λ of pairs (g,J) where $J \subset I$ and, setting $V_J = \bigcup_{j \in J} \bar{V}_j$, we have $g \in G(V_J)$ and $b(g) = s|V_J$. Now $\Lambda \neq \emptyset$ and is partially ordered by inclusion. The requirement's of Zorn's lemma are clearly satisfied and so Λ contains a maximal element, say (t,K). It is suffiicent to show $K = I$. Suppose $i \in I \setminus K$. Then $b(t - t_i) = 0$ on $V(K) \cap \bar{V}_i$ and so there exists $r_i \in F(V(K) \cap \bar{V}_i)$ such that $t - t_i = a(r_i)$. Since F is soft, r_i extends to U_i. But now $(t,K) \cup (a(r_i) + t_i, i)$ extends (t,K) contradicting the maximality of (t,K). Therefore $K = I$. □

Corollary 6.3.3. Let $0 \to F \xrightarrow{a} G \xrightarrow{b} H \to 0$ be a short exact sequence of sheaves over X. If F and G are soft so is H.

Proof. Let $K \subset X$ be closed. We must show that every $s \in H(K)$ extends to a section of H over X. First observe that F_K is soft. Hence $0 \to F_K \xrightarrow{a} G_K \xrightarrow{b} H_K \to 0$ is a short exact sequence of sheaves for which F_K is soft and, applying Proposition 6.3.2, there exists a section t of G over K such that $b(t) = s$. Since G is soft, t extends to a section of G over X. □

Corollary 6.3.4. Let $0 \to F_0 \xrightarrow{a_0} F_1 \xrightarrow{a_1} F_2 \to \cdots$ be a long exact sequence of soft sheaves over X. Then the corresponding complex

$$0 \to F_0(X) \xrightarrow{a_0} F_1(X) \xrightarrow{a_1} F_2(X) \xrightarrow{a_2} \cdots$$

is exact.

Proof. For $i \geq 0$, let K_i denote the sheaf $\mathrm{Ker}(a_i)$. The exactness of the given long exact sequence is then equivalent to the exactness of the short exact sequences

$$0 \to K_i \to F_i \xrightarrow{a_{i+1}} K_{i+1} \to 0, \quad i \geq 0 .$$

For $i = 0$, $K_0 = F_0$ and so is soft. Hence, by Corollary 6.3.3, K_1 is soft. By induction, every K_i is soft. Hence by Proposition 6.3.2, the sequences

$$0 \to K_i(X) \to F_i(X) \xrightarrow{a_{i+1}} K_{i+1}(X) \to 0$$

are all exact. But this is equivalent to the required result. □

Definition 6.3.5. A sheaf (F,π,X) is *fine* if it admits a partition of unity of the identity morphism of F subordinate to any locally finite open cover of X. That is, given a locally finite open cover $\{U_i\}$ of X, there exist sheaf morphisms $\eta_i\colon F \to F$ satisfying

1. $\eta_i = 0$ outside of some closed subset of X contained in U_i.

2. $\sum_{i \in I} \eta_i = I$, the identity morphism of F.

Proposition 6.3.6. Every fine sheaf is soft.

Proof. Let F be a fine sheaf over X, $K \subset X$ be closed and $\tilde{s} \in F(K)$. By Lemma 6.1.12, we may find an open neighbourhood U of K in X and $\tilde{s} \in F(U)$ extending s. Take the open cover $\{U, X \setminus K\}$ of X and a partition of unity $\{\theta,\eta\}$ of the identity morphism of F subordinate to this cover. Observe that $\theta | K = I$. Setting $\tilde{s} = 0$ outside U, we see that $\theta\tilde{s}$ is the required section of F. □

Examples.

1. Let E be a smooth vector bundle over the differential manifold X. Then $\underline{E}_\infty$ is fine. Indeed, suppose that $\{U_i\}$ is any locally finite open cover of X and let $\{\eta_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. The η_i induce sheaf morphisms $\eta_i\colon \underline{E}_\infty \to \underline{E}_\infty$ and clearly $\{\underline{\eta}_i)$ is a partition of unity of the identity morphism of $\underline{E}_\infty$ which is subordinate to $\{U_i\}$. Hence $\underline{E}_\infty$ is fine. Consequently all the sheaves $\underline{C}^p$, $\underline{C}^{p,q}$ that we have defined on differential and complex manifolds are fine and, therefore, soft. On the other hand, if $\underline{E}$ is the sheaf of holomorphic sections of a holomorphic vector bundle E then $\underline{E}$ is never soft.

2. Let F be a sheaf on X. Let F^* denote the sheaf of germs of not necessarily continuous sections of F. Thus, $F^*(U)$ will be the set of *all* sections of F over U. The sheaf F^* is is obviously fine. In fact it is also *flabby*: Every section of F^* over an open subset of X extends

to X. Flabby sheaves are used in the development of sheaf cohomology in algebraic geometry where the spaces are not even Hausdorff, let alone paracompact (see also the exercises at the end of this section).

We need one more definition before we can define the sheaf cohomology groups of a sheaf on X.

Definition 6.3.7. Let F be a sheaf on X. A *resolution* of F is a long exact sequence

$$0 \to F \to F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} F_2 \xrightarrow{d_2} \cdots$$

A resolution of F will be called soft (resp. fine) if each of the sheaves F_i is soft (resp. fine).

Proposition 6.3.8. Every sheaf F on X has a fine resolution.

Proof. Let F^* denote the sheaf of germs of not necessarily continuous sections of F (example 2 above) and let $\varepsilon: F \to F^*$ denote the corresponding inclusion map. Set $F_0 = F^*$ and let $\bar{F}_0 = F^*/F$. We have the short exact sequence $0 \to F \to F_0 \xrightarrow{q_0} \bar{F}_0 \to 0$. Proceeding inductively, let $F_{j+1} = (\bar{F}_j)^*$ and $\bar{F}_{j+1} = F_{j+1}/F^*_{j+1}$. For $j \geq 0$ we have the short exact sequences

$$0 \to \bar{F}_j \xrightarrow{\varepsilon_{j+1}} F_{j+1} \xrightarrow{q_{j+1}} \bar{F}_{j+1} \to 0 .$$

Here, of course, $\bar{F}_0 = F$. Hence we have the corresponding long exact sequence

$$0 \to F \xrightarrow{\varepsilon} F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \cdots$$

where $d_j = \varepsilon_{j+1}q_j$ and $\varepsilon = \varepsilon_0$. Since the sheaves F_j are all fine, we have constructed a fine resolution of F. □

Remark. We call the resolution of F constructed in the proof of Proposition 6.3.8 the *canonical resolution* of F.

Let F be a sheaf on X and $0 \to F \xrightarrow{\varepsilon} F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \cdots$ be the canonical resolution of F. Associated to the canonical resolution of F we have the complex

$$0 \to F(X) \xrightarrow{\varepsilon^*} F_0 \xrightarrow{d_0^*} F_1 \xrightarrow{d_1^*} \cdots \ .$$

We set

$$H^0(X,F) = \mathrm{Ker}(d_0^*); \ H^p(X,F) = \mathrm{Ker}(d_p^*)/\mathrm{Im}(d_{p-1}^*), \ p > 0 \ .$$

Each $H^p(X,F)$ is an abelian group.

Definition 6.3.9. The group $H^p(X,F)$ constructed above is called the *pth. sheaf cohomology group of X with coefficients in the sheaf* F.

Theorem 6.3.10. The sheaf cohomology groups of X satisfy the following basic properties:

1. Given a sheaf F on X then

   A) $H^0(X,F) \approx F(X)$.

   B) $H^p(X,F) = 0$ for $p > 0$ if F is fine.

2. If $a: F \to G$ is a morphism of sheaves over X then for $p \geq 0$ there are induced homomorphisms $a^p: H^p(X,F) \to H^p(X,G)$ satisfying

   A) $a^0: H^0(X,F) \to H^0(X,G)$ is precisely the map on sections induced by a.

   B) If $a: F \to F$ is the identity, then a^p is the identity, $p \geq 0$.

   C) If $a: F \to G$, $b: G \to H$ are morphisms of sheaves over X then for $p \geq 0$ we have $(ba)^p = b^p a^p: H^p(X,F) \to H^p(X,H)$.

3. If $0 \to F \xrightarrow{a} G \xrightarrow{b} H \to 0$ is a short exact sequence of sheaves on X then for $p \geq 0$ there is a connecting homomorphism $\delta: H^p(X,H) \to H^{p+1}(X,F)$ satisfying

   A) The cohomology sequence

$$0 \to H^0(X,F) \xrightarrow{a^0} H^0(X,G) \xrightarrow{b^0} H^0(X,H) \xrightarrow{\delta} H^1(X,F) \xrightarrow{a^1} \cdots$$

   is exact.

   B) Given a commutative diagram of short exact sequences on X

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{a} & G & \xrightarrow{b} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \omega} & & \\ 0 & \longrightarrow & A & \xrightarrow{c} & B & \xrightarrow{d} & C & \longrightarrow & 0 \end{array}$$

the corresponding cohomology diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0(X,F) & \xrightarrow{a^0} & H^0(X,G) & \xrightarrow{b^0} & H^0(X,H) & \xrightarrow{\delta} & H^1(X,F) & \xrightarrow{a^1} & \cdots \\
 & & \downarrow \eta^0 & & \downarrow \phi^0 & & \downarrow \omega^0 & & \downarrow \eta^1 & & \\
0 & \longrightarrow & H^0(X,A) & \xrightarrow{c^0} & H^0(X,B) & \xrightarrow{d^0} & H^0(X,C) & \xrightarrow{\delta} & H^1(X,A) & \xrightarrow{c^1} & \cdots
\end{array}$$

commutes.

Proof. 1A is obvious and 1B follows from Corollary 6.3.4. In our proof of 2 we follow the notational conventions of the proof of Proposition 6.3.8. Observe that a morphism $a\colon F \to G$ induces a morphism $a^0\colon F_0 \to G_0$ and so a morphism $\bar{a}^0\colon \bar{F}_0 \to \bar{G}_0$. Clearly the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & F_0 & \longrightarrow & \bar{F}_0 & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow a_0 & & \downarrow \bar{a}_0 & & \\
0 & \longrightarrow & G & \longrightarrow & G_0 & \longrightarrow & \bar{G}_0 & \longrightarrow & 0
\end{array}$$

commutes. Proceeding inductively, we see that for $j \geq 0$ we have morphisms $a^j\colon F_j \to G_j, \bar{a}_j\colon \bar{F}_j \to \bar{G}_j$ such that the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{F}_j & \longrightarrow & F_j & \longrightarrow & \bar{F}_{j+1} & \longrightarrow & 0 \\
 & & \downarrow \bar{a}^j & & \downarrow a^j & & \downarrow \bar{a}^{j+1} & & \\
0 & \longrightarrow & \bar{G}_j & \longrightarrow & G_j & \longrightarrow & \bar{G}_{j+1} & \longrightarrow & 0
\end{array}$$

commute. Hence we have the commutative ladder of long exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\varepsilon} & F_0 & \xrightarrow{d_0} & F_1 & \xrightarrow{d_1} & \cdots \\
 & & \downarrow a & & \downarrow a^0 & & \downarrow a^1 & & \\
0 & \longrightarrow & G & \xrightarrow{\varepsilon} & G_0 & \xrightarrow{d_0} & G_1 & \xrightarrow{d_1} & \cdots
\end{array}$$

and hence the corresponding commutative diagraom of sections:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(X) & \xrightarrow{\varepsilon} & F_0(X) & \xrightarrow{d_0^*} & F_1(X) & \xrightarrow{d_1^*} & \cdots \\
 & & \downarrow a & & \downarrow a^0 & & \downarrow a^1 & & \\
0 & \longrightarrow & G(X) & \xrightarrow{\varepsilon} & G_0(X) & \xrightarrow{d_0^*} & G_1(X) & \xrightarrow{d_1^*} & \cdots
\end{array}$$

Suppose $s \in \operatorname{Ker}(d_j^*)$. The commutativity of the diagram implies that $d_j^*(a^j s) = a^j d_j^* s = 0$. Hence $a^j(\operatorname{Ker}(d_j^*)) \subset \operatorname{Ker}(d_j^*)$. Similarly, $a^j(\operatorname{Im}(d_{j-1}^*)) \subset \operatorname{Im}(d_{j-1}^*)$. Therefore, a^j induces a homomorphism $a^j\colon H^j(X,F) \to H^j(X,G)$. Properties 2A, B, C all follow straightforwardly from the definition of the induced maps on cohomology and the naturality of our constructions and we leave their verification to the reader.

It remains to prove 3. Take canonical resolutions of the sheaves F, G, H. As in the proof of 2 we have an induced sequence between these resolutions and a corresponding commutative diagram of sections, a typical portion of which is displayed below.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_{j-1}(X) & \xrightarrow{a^{j-1}} & G_{j-1}(X) & \xrightarrow{b^{j-1}} & H_{j-1}(X) & \longrightarrow & 0 \\
 & & \downarrow d_{j-1} & & \downarrow d_{j-1} & & \downarrow d_{j-1} & & \\
0 & \longrightarrow & F_j(X) & \xrightarrow{a^j} & G_j(X) & \xrightarrow{b^j} & H_j(X) & \longrightarrow & 0 \\
 & & \downarrow d_j & & \downarrow d_j & & \downarrow d_j & & \\
0 & \longrightarrow & F_{j+1}(X) & \xrightarrow{a^{j+1}} & G_{j+1}(X) & \xrightarrow{b^{j+1}} & H_{j+1}(X) & \longrightarrow & 0 \\
 & & \downarrow d_{j+1} & & \downarrow d_{j+1} & & \downarrow d_{j+1} & & \\
0 & \longrightarrow & F_{j+2}(X) & \xrightarrow{a^{j+2}} & G_{j+2}(X) & \xrightarrow{b^{j+2}} & H_{j+2}(X) & \longrightarrow & 0
\end{array}
$$

Observe that the rows in the diagram are all exact since the sheaves F_j, G_j and H_j are all soft. We now show how to define the connecting homomorphism $\delta\colon H^j(X,H) \to H^{j+1}(X,F)$. Let $\alpha \in H^j(X,H)$ and $H \in \operatorname{Ker}(d_j) \subset H_j(X)$ be a representative for α. Now $H = b^j(G)$ for some $G \in G_j(X)$. Since $d_j b^j = b^{j+1} d_j$, we see that $b^{j+1} d_j(G) = 0$ and so there exists $F \in F_{j+1}(X)$ such that $a^{j+1}(F) = d_j(G)$. But $a^{j+2} d_{j+1}(F) = d_{j+1} a^{j+1}(F) = d_{j+1} d_j(G) = 0$. Since a^{j+2} is injective it follows that $d_{j+1}(F) = 0$ and so F defines an element of $H^{j+1}(X,F)$. We claim that the class of F in $H^{j+1}(X,F)$ depends only on α and not on our choices of H, G and F. Granted this, our construction defines the connecting homomorphism $\delta\colon H^j(X,H) \to H^{j+1}(X,F)$. Suppose that H', G' and F' are the result of another sequence of choices. Since $H - H'$ defines the zero element of $H^j(X,H)$, there exists $\tilde{H} \in H_{j-1}(X)$ such that $H - H' = d_{j-1}(\tilde{H})$. Certainly $\tilde{H} = b^{j-1}(\tilde{G})$ for some $\tilde{G} \in G_{j-1}(X)$ and so $b^j(G - G' - d_{j-1}(\tilde{G})) = 0$. Hence $G - G' - d_{j-1}\tilde{G} = a^j\tilde{F}$, for some $\tilde{F} \in F_j(X)$.

Now $a^{j+1}d_j(F) = d_ja^j(F) = d_j(G) - d_j(G') = a^{j+1}(F) - a^{j+1}(F') = a^{j+1}(F - F')$ and so, since a^{j+1} is injective, we have $F - F' = d_j(F)$. Therefore, F and F' define the same class in $H^{j+1}(X,\mathcal{F})$.

The proof of 3A involves a straightforward diagram chase and we leave details to the reader. To prove 3B we take canonical resolutions of the sequences and form the corresponding 3-dimensional commutative diagram of sequences of sections. Given $\alpha \in H^j(X,\mathcal{H})$, we construct, as above, elements H, G, F such that the cohomology classes of H and F define α and $\delta\alpha$ respectively. It now suffices to observe that the class of $\omega^j H$ is $\eta^j\alpha$ and that $\omega^j H$, $\phi^j G$, $\eta^{j+1}F$ is a sequence defining $\delta(\omega^j\alpha)$. Hence $\delta\omega^j = \omega^{j+1}\delta$. The commutativity of the squares not involving δ is, of course, immediate from 2C. □

Theorem 6.3.11. Let $0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_2 \xrightarrow{d_2} \dots$ be a resolution of the sheaf $\mathcal{F}$. Suppose that $H^j(X,\mathcal{F}_k) = 0$, $j > 0$, $k \geq 0$. Then

$$H^0(X,\mathcal{F}) \cong \mathrm{Ker}(d_0^*); \quad H^p(X,\mathcal{F}) \cong \mathrm{Ker}(d_p^*)/\mathrm{Im}(d_{p-1}^*), \ p > 0.$$

Proof. Let $\mathcal{K}_j = \mathrm{Ker}(d_j)$, $j \geq 0$. The exactness of the resolution of $\mathcal{F}$ is equivalent to the exactness of the sequences

$$0 \to \mathcal{K}_j \to \mathcal{F}_j \xrightarrow{d_j} \mathcal{K}_{j+1} \to 0, \ j \geq 0 .$$

Take the long exact cohomology sequences of these short exact sequences:

$$\longrightarrow H^p(X,\mathcal{F}_j) \to H^p(X,\mathcal{K}_{j+1}) \xrightarrow{\delta} H^{p+1}(X,\mathcal{K}_j) \to H^{p+1}(X,\mathcal{F}_j) \longrightarrow \quad .$$

Since $H^p(X,\mathcal{F}_j) = 0$, $p > 0$, we see that

A.... $$H^p(X,\mathcal{K}_{j+1}) \cong H^{p+1}(X,\mathcal{K}_j), \ p > 0, \ j \geq 0 .$$

We also have the initial portions of the long exact cohomology sequences:

$$\longrightarrow H^0(X,\mathcal{F}_j) \xrightarrow{d_j^*} H^0(X,\mathcal{K}_{j+1}) \xrightarrow{\delta} H^1(X,\mathcal{K}_j) \to 0$$

and so $H^1(X,\mathcal{K}_j) \cong H^0(X,\mathcal{K}_{j+1})/d_j^*H^0(X,\mathcal{F}_j)$. Now $H^0(X,\mathcal{K}_{j+1}) = \mathrm{Ker}(d_{j+1}^*)$ and so

B.... $H^1(X,K_j) \cong Ker(d^*_{j+1})/Im(d^*_j)$, $j \geq 0$.

By repeated application of A and one application of B we see that for $p > 0$

$$H^p(X,F) = H^p(X,K_0) \cong H^{p-1}(X,K_1) \cong \ldots \cong H^1(X,K_{p-1}) \cong Ker(d^*_p)/Im(d^*_{p-1}) .$$

Finally the result for $p = 0$ follows from the exactness of the sequence

$$0 \to H^0(X,F) \xrightarrow{\varepsilon^*} H^0(X,F_0) \to H^0(X,K_1) . \qquad \square$$

As an immediate consequence of Theorem 6.3.11 together with 1B of Theorem 6.3.10 we have

Corollary 6.3.12. We can compute the cohomology of X with coefficients in F using any fine (or soft) resolution of F. In particular, Properties 1 - 3 of Theorem 6.3.10 determine the cohomology groups $H^j(X,F)$ up to isomorphism.

Remark. Suppose that $\tilde{H}^j(X,F)$, $j \geq 0$, are the groups of another sheaf cohomology theory for X. That is, we suppose that the groups $\tilde{H}^j(X,F)$ satisfy all the properties listed in Theorem 6.3.10. By Corollary 6.3.12, we have $\tilde{H}^j(X,F) \cong H^j(X,F)$ for all sheaves F on X. It can be shown that this isomorphism is natural - in particular, commutes with connecting homomorphisms. For the general proof the reader may consult Godement [1]. We shall prove the existence of this natural isomorphism for the case of Čech cohomology later in the section.

Examples.

3. Let X be a topological manifold. Taking the canonical resolution of the constant sheaf $\mathbb{Z}$ we may define the cohomology groups $H^p(X, \mathbb{Z})$, $p \geq 0$. We may also define the singular cohomology groups $H^p_{sing}(X, \mathbb{Z})$ of X (see Spanier [1], Greenberg [1]). We claim that $H^p(X, \mathbb{Z}) \cong H^p_{sing}(X, \mathbb{Z})$, $p \geq 0$. First, let $S_p(U, \mathbb{Z})$ denote the abelian groups of integral singular p-chains on the open subset U of X. Then $S^p(U, \mathbb{Z}) = Hom_{\mathbb{Z}}(S_p(U, \mathbb{Z}), \mathbb{Z})$ is the group of singular integral p-cochains on U. We let $D = D_p$: $S^p(U,\mathbb{Z}) \to S^{p+1}(U, \mathbb{Z})$ denote the coboundary

homomorphism. The assignment $U \to S^p(U,\mathbb{Z})$ defines a presheaf $S^p(\mathbb{Z})$ on X and we let $\underline{S}^p(\mathbb{Z})$ denote the corresponding sheaf, $p \geq 0$. Since D commutes with restrictions, we arrive at the sheaf complex

A.... $$0 \to \mathbb{Z} \to \underline{S}^0(\mathbb{Z}) \xrightarrow{D_0} \underline{S}^1(\mathbb{Z}) \xrightarrow{D_1} \ldots$$

By definition,

$$H^0_{sing}(X,\mathbb{Z}) = \mathrm{Ker}(D_0^*)$$

$$H^0_{sing}(X,\mathbb{Z}) = \mathrm{Ker}(D_p^*)/\mathrm{Im}(D_{p-1}^*),\ p \geq 1.$$

We claim that (A) is a fine resolution of $\mathbb{Z}$. Exactness follows since every point of X has a neighbourhood base of contractable open neighbourhoods which have vanishing singular cohomology by standard theory. To prove fineness of the sheaves $\underline{S}^j(\mathbb{Z})$ first observe that $\underline{S}^0(\mathbb{Z}) = \mathbb{Z}^*$ - sheaf of germs of discontinuous sections of $\mathbb{Z}$. Hence $\underline{S}^0(\mathbb{Z})$ is fine. Suppose that $p > 0$ and $s \in \underline{S}^p(\mathbb{Z})(K)$, $K \subset X$ closed. By Lemma 6.1.12, s is the restriction of a continuous section $\tilde{s}$ of $\underline{S}^p(\mathbb{Z})$ over some open neighbourhood U of K. Let 1_K be the (continuous!) section of $\underline{S}^0(\mathbb{Z})$ over X defined by $1_K|K \equiv 1$, $1_K|X \setminus K \equiv 0$. Since $\underline{S}^p(\mathbb{Z})$ is a sheaf of $\underline{S}^0(\mathbb{Z})$-modules, $1_K\tilde{s}$ is a continuous section of $\underline{S}^p(\mathbb{Z})$ over X extending s. We may now apply Corollary 6.3.12 to obtain the required isomorphisms between $H^p(X,\mathbb{Z})$ and $H^p_{sing}(X,\mathbb{Z})$. We conclude this example by making two additional remarks. First, replacing $\mathbb{Z}$ by any abelian group G (or commutative ring with 1), we can repeat the proof to obtain isomorphisms between $H^p(X,G)$ and $H^p_{sing}(X,G)$. Secondly, if X is a differential manifold we may define the sheaves $\underline{S}^p_\infty(\mathbb{Z})$ of smooth (that is C^∞) p-cochains on X. It is a basic and well known fact in differential topology that the complex $0 \to \mathbb{Z} \to \underline{S}^0_\infty(Z) \xrightarrow{D_0} \underline{S}^1_\infty(Z) \to \ldots$ continues to define the singular cohomology groups of X. In particular, the complex is a fine resolution of $\mathbb{Z}$.

4. Let X be an n-dimensional differential manifold and $0 \to \mathbb{C} \to \underline{C}^0 \xrightarrow{d} \ldots \xrightarrow{d} \underline{C}^n \to 0$ be the de Rham complex of X. Since the sheaves $\underline{C}^p$ are all fine, the de Rham complex is a fine resolution of the constant sheaf $\mathbb{C}$. Let

$$H^0_{DR}(X,\mathbb{C}) = \text{Ker } d\colon C^0(X) \to C^1(X)$$

$$H^p_{DR}(X,\mathbb{C}) = \frac{\text{Ker } d\colon C^p(X) \to C^{p+1}(X)}{\text{Im } d\colon C^{p-1} \to C^p(X)}, \quad p \geq 1.$$

denote the *de Rham cohomology* groups of X. Corollary 6.3.12 implies that

$$H^p_{DR}(X,\mathbb{C}) \cong H^p(X,\mathbb{C}), \quad p \geq 0.$$

In particular, $H^p(X,\mathbb{C}) = 0$, $p > n$.
We showed in Example 3 that $H^p(X,\mathbb{C}) \cong H^p_{sing}(X,\mathbb{C})$ and we shall now describe an explicit isomorphism between $H^p_{DR}(X,\mathbb{C})$ and $H^p_{sing}(X,\mathbb{C})$. For $p \geq 0$ we define a sheaf morphism $I^p\colon \underline{C}^p \to \underline{S}^p(\mathbb{C})$ by integration of chains:

$$I^p(f)(c) = \int_c f, \quad f \in C^p(U),\ c \in S_p(U,\mathbb{C}).$$

By Stokes' theorem, we see that I^p commutes with the differentials d, D and so we obtain a commutative ladder of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \underline{C}^0 & \xrightarrow{d} & \underline{C}^1 & \xrightarrow{d} & \cdots \\ & & \updownarrow I & & \downarrow I^0 & & \downarrow I^1 & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & S^0(\mathbb{C}) & \xrightarrow{D} & S^1(\mathbb{C}) & \xrightarrow{D} & \cdots \end{array}$$

The morphisms I^p induce the de Rham isomorphisms between $H^p_{DR}(X,\mathbb{C})$ and $H^p_{sing}(X,\mathbb{C})$.
Finally we should point out that the de Rham isomorphisms actually give an *algebra* isomorphism between the de Rham and singular cohomology groups (the algebra structure on the de Rham and singular cohomology groups is given by wedge and cup product respectively). The reader may find a proof of this stronger statement in Warner [1].

5. Let X be an n-dimensional complex manifold. For $p \geq 0$, we have the Dolbeault complex

$$0 \to \Omega^p \to \underline{C}^{p,0} \xrightarrow{\bar\partial} \underline{C}^{p,1} \xrightarrow{\bar\partial} \cdots \longrightarrow \underline{C}^{p,n} \to 0.$$

Since the sheaves $\underline{C}^{p,q}$ are fine, $p,q \geq 0$, we therefore obtain the *Dolbeault isomorphisms*

$$H^q(X,\Omega^p) \cong \frac{\text{Ker } \bar{\partial}: C^{p,q}(X) \to C^{p,q+1}(X)}{\text{Im } \bar{\partial}: C^{p,q-1}(X) \to C^{p,q}(X)}, \quad q \geq 0.$$

In particular,

$$H^q(X,\mathcal{O}) = \frac{\text{Ker } \bar{\partial}: C^{0,q}(X) \to C^{0,q+1}(X)}{\text{Im } \bar{\partial}: C^{0,q-1}(X) \to C^{0,q}(X)}, \quad q \geq 0 .$$

If E is a holomorphic vector bundle on X, we have the Dolbeault complex

$$0 \to \Omega^p(\underline{E}) \to \underline{C}^{p,0}(E) \xrightarrow{\bar{\partial}} C^{p,1}(E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} C^{p,n}(E) \to 0$$

and corresponding Dolbeault isomorphisms

$$H^q(X,\Omega^p(\underline{E})) \cong \frac{\text{Ker } \bar{\partial}: C^{p,q}(M,E) \to C^{p,q+1}(M,E)}{\text{Im } \bar{\partial}: C^{p,q-1}(M,E) \to C^{p,q}(M,E)}, \quad p,q \geq 0 .$$

6. We have the short exact sequence $0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{M}/\mathcal{O} \to 0$ of sheaves over any complex manifold X. Take the initial portion of the long exact cohomology sequence:

$$\begin{array}{ccccccc} 0 \to & H^0(X,\mathcal{O}) & \to H^0(X,\mathcal{M}) & \to & H^0(X,\mathcal{M}/\mathcal{O}) & \xrightarrow{\delta} & H^1(X,\mathcal{O}) \\ & \| & & & \| & & \\ & A(X) & & & \mathcal{M}(X) & & \end{array}$$

Given the data $\alpha \in H^0(X,\mathcal{M}/\mathcal{O})$ for the Cousin I problem on X, we see that we can solve the Cousin I problem for α if and only if $\delta\alpha = 0$ in $H^1(X,\mathcal{O})$. In particular, X will be a Cousin I domain if and only if $\text{Im } \delta = 0$ in $H^1(X,\mathcal{O})$. Later we shall see that $H^1(X,\mathcal{O}) = 0$ whenever X is a Stein manifold and so Stein manifolds are Cousin I domains.

7. Let Z be an analytic subset of the complex manifold X. We have the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

of sheaves over X and corresponding initial portion of the long exact cohomology sequence:

$$\longrightarrow H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_Z) \xrightarrow{\delta} H^1(X,\mathcal{I}_Z) \longrightarrow \ldots$$

$$\begin{array}{cc} \| & \| \\ A(X) & A(Z) \end{array}$$

Suppose $f \in A(Z)$. Then f is the restriction of an analytic function on X if and only if $\delta f = 0$. In particular, if $H^1(X,\mathcal{I}_Z) = 0$ every analytic function on Z extends to an analytic function on X. We shall see later that this cohomology group vanishes whenever X is Stein.

Examples 6 and 7 above should convince the reader of the importance of computing sheaf cohomology groups in complex analysis.

For the remainder of this section we develop Čech cohomology theory for sheaves over a paracompact space. As we shall see Čech theory is computable - at least for all the examples that interest us - and is also isomorphic to the sheaf cohomology theory we have already constructed.

Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X and p be a non-negative integer. Given $s = (s_0,\ldots,s_p) \in I^{p+1}$, set $U_s = U_{s_0} \cap \ldots \cap U_{s_p}$. A *p-cochain* of $\mathcal{U}$ with values in $\mathcal{F}$ is a map c which assigns to each $s \in I^{p+1}$ a section $c_s \in \mathcal{F}(U_s)$ and for which c_s is an alternating function of s. That is, $c_{s_0\ldots s_i \ldots s_j \ldots s_p} = -c_{s_0 \ldots s_j \ldots s_i \ldots s_p}$, $0 \le i < j \le p$. We let $C^p(\mathcal{U},\mathcal{F})$ denote the abelian group of all p-cochains of $\mathcal{U}$ with values in $\mathcal{F}$.

For $p \ge 0$, we define a coboundary operator $D: C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ by

$$(Dc)_s = \sum_{j=0}^{p+1} (-1)^j c_{s_0\ldots\hat{s}_j\ldots s_{p+1}}, \quad s \in I^{p+2}.$$

A simple computation shows that $D^2 = 0$. We let

$$Z^p(\mathcal{U},\mathcal{F}) = \{c \in C^p(\mathcal{U},\mathcal{F}): Dc = 0\}; \quad B^p(\mathcal{U},\mathcal{F}) = \{Dc: c \in C^{p-1}(\mathcal{U},\mathcal{F})\}$$

denote the groups of *p-cocycles* and *p-coboundaries* respectively. Here we take $C^{-1}(\mathcal{U},\mathcal{F}) = 0$ and $B^0(\mathcal{U},\mathcal{F}) = 0$. Since $D^2 = 0$, $B^p(\mathcal{U},\mathcal{F})$ is a subgroup of $Z^p(\mathcal{U},\mathcal{F})$ and we let $H^p(\mathcal{U},\mathcal{F})$ denote the quotient group $Z^p(\mathcal{U},\mathcal{F})/B^p(\mathcal{U},\mathcal{F})$. We call $H^p(\mathcal{U},\mathcal{F})$ the *pth. cohomology group of* $\mathcal{U}$ *with values in* $\mathcal{F}$.

Example 8. The data for the Cousin A (resp. Cousin B) problem on a complex manifold defines a class in $H^1(\mathcal{U},\mathcal{O})$ (resp. $H^1(\mathcal{U},\mathcal{O}^*)$).

Lemma 6.3.13. For any sheaf F on X we have

$$H^0(\mathcal{U},F) \approx F(X).$$

Proof. Certainly $Z^0(\mathcal{U},F) = H^0(\mathcal{U},F)$. But if $c \in Z^0(\mathcal{U},F)$, $c_i - c_j = 0$ on U_{ij} for all $i,j \in I$. Hence we may define $f \in F(X)$ by $f|U_i = c_i$. □

Lemma 6.3.14. Let $a: F \to G$ be a morphism of sheaves over X. For $p \geq 0$ we have induced maps $a^p_{\mathcal{U}}: H^p(\mathcal{U},F) \to H^p(\mathcal{U},G)$ satisfying

A. $a^0_{\mathcal{U}}: H^0(\mathcal{U},F) \to H^0(\mathcal{U},G)$ is precisely the map on sections induced by a.

B. If a is the identity map of F, then $a^p_{\mathcal{U}}$ is the identity, $p \geq 0$.

C. If $a: F \to G$, $b: G \to H$ are morphisms of sheaves over X then $(ba)^0_{\mathcal{U}} = b^p_{\mathcal{U}} a^p_{\mathcal{U}}$, $p \geq 0$.

Proof. Elementary and left to the reader. □

Lemma 6.3.15. Suppose that F is fine. Then $H^p(\mathcal{U};F) = 0$, $p > 0$.

Proof. Let $\mathcal{V} = \{V_j, j \in J\}$ be a locally finite open refinement of $\mathcal{U}$ chosen such that there is a map $\phi: J \to I$ with $\bar{V}_j \subset U_{\phi(j)}$, $j \in J$. Choose a partition of unity $\{\eta_j\}$ for the sheaf F which is subordinate to $\mathcal{V}$. Let $c \in Z^p(\mathcal{U},F)$. For $j \in J$, define $b^j \in C^{p-1}(\mathcal{U},F)$ by

$$\begin{aligned} b^j_s &= 0 \text{ if } U_s \cap V_j = \emptyset \\ &= (-1)^p \eta_j c_{s_0 \ldots s_{p-1}\phi(j)} \text{ if } U_s \cap V_j \neq \emptyset . \end{aligned}$$

(Here $\eta_j c_{s_0 \ldots \phi(j)}$ is defined to be zero outside V_j). A simple computation shows that for all $s \in I^{p+1}$ we have

$$(Db^j)_s = \eta_j c_s.$$

Set $b = \sum_{j \in J} b^j$. Since $\{\eta_j\}$ is a partition of unity, we see that

$DB = c$. Hence $H^p(\mathcal{U}, F) = 0$. □

Remark. To prove this result we needed to know that F was fine and not just soft. It is for this reason that we put "fine" rather than "soft" in 1B of Theorem 6.3.10. See also Godement [1; Theorem 5.2.3, Chapter 2].

Theorem 6.3.16. Let F be a sheaf on X and $\mathcal{U}$ be an open cover of X. For $p \geq 0$ we have a canonical homomorphism $p(\mathcal{U})\colon H^p(\mathcal{U}, F) \to H^p(X, F)$ satisfying

A. If $a\colon F \to G$ is a homomorphism then

$$\begin{array}{ccc} H^p(\mathcal{U},F) & \xrightarrow{p(\mathcal{U})} & H^p(X,F) \\ \big\downarrow a^p_{\mathcal{U}} & & \big\downarrow a^p \\ H^p(\mathcal{U},G) & \xrightarrow{p(\mathcal{U})} & H^p(X,G) \end{array}$$

commutes.

B. If $H^p(U_s, F) = 0$ for all $s \in I^{p+1}$ and $p > 0$, then $p(\mathcal{U})$ is an isomorphism (*Leray's Theorem*).

Proof. Undoubtedly the most elegant proof of this result uses spectral sequences - see, for example, Godement [1]. Our proof is an elementary diagram chase:

Set $F_{-1} = F$ and let $0 \to F_{-1} \xrightarrow{d_{-1}} F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \ldots$ denote the canonical resolution of F. For $s \in I^{p+1}$ we have the sequence of sections

$$0 \to F_{-1}(U_s) \xrightarrow{d_{-1}} F_0(U_s) \xrightarrow{d_0} \ldots$$

and so, taking the direct sum over I^{p+1}, we have for $p \geq 0$ sequences

A.... $$0 \to C^p(\mathcal{U}, F_{-1}) \xrightarrow{d_{-1}} C^p(\mathcal{U}, F_0) \xrightarrow{d_0} \ldots$$

Associated to each of the sheaves F_j, $j \geq -1$, we have sequences

$$\text{B....} \qquad 0 \to F_j(X) \xrightarrow{D_{-1}} C^0(\mathcal{U},F_j) \xrightarrow{D_0} C^1(\mathcal{U},F_j) \to \ldots$$

Here D_{-1} denotes the inclusion map and D_j is the appropriate coboundary operator for $j \geq 0$. Since F_j is fine, $j \geq 0$, the sequences B are exact for $j \geq 0$. Combining the sequences A and B we arrive at a commutative diagram of sequences the initial portion of which is displayed below.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{-1}(X) & \xrightarrow{d_{-1}} & F_0(X) & \xrightarrow{d_0} & F_1(X) & \xrightarrow{d_1} & \ldots \\
 & & \downarrow D_{-1} & & \downarrow D_{-1} & & \downarrow D_{-1} & & \\
0 & \longrightarrow & C^0(\mathcal{U},F_{-1}) & \xrightarrow{d_{-1}} & C^0(\mathcal{U},F_0) & \xrightarrow{d_0} & C^0(\mathcal{U},F_1) & \xrightarrow{d_1} & \ldots \\
 & & \downarrow D_0 & & \downarrow D_0 & & \downarrow D_0 & & \\
0 & \longrightarrow & C^1(\mathcal{U},F_{-1}) & \xrightarrow{d_{-1}} & C^1(\mathcal{U},F_0) & \xrightarrow{d_0} & C^1(\mathcal{U},F_1) & \xrightarrow{d_1} & \ldots \\
 & & \downarrow D_1 & & \downarrow D_1 & & \downarrow D_1 & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The columns of this diagram are all exact, save the first. We now show how to construct the required homomorphism $p(\mathcal{U})$: $H^p(\mathcal{U},F) \to H^p(X,F)$. Let $c_0 \in Z^p(\mathcal{U},F)$ represent the class $\alpha \in H^p(\mathcal{U},F)$. We construct inductively a sequence $c_j \in C^{p-j}(\mathcal{U},F_{j-1})$ which satisfies

$$d_{p-j}c_j = D_{p-j+1}c_{j+1}, \quad 0 \leq j \leq p.$$

Suppose that we have constructed c_j, $j \leq r < p+1$. Then $D_{p-r}d_{r-1}c_r = d_{r-1}D_{p-r}c_r = d_{r-1}d_{r-2}c_{r-1} = 0$. Hence by exactness of the $(r+1)$th. column, there exists $c_{r+1} \in C^{p-r-1}(\mathcal{U},F_r)$ such that $D_{p-r-1}c_{r+1} = d_{r-1}c_r$. This completes the inductive step. Now $c_{p+1} \in F_p(X)$. Observe that $D_{-1}d_pc_{p+1} = d_pD_{-1}c_{p+1} = d_pd_{p-1}c_p = 0$ and so, since D_{-1} is injective, $d_pc_{p+1} = 0$. Therefore $c_{p+1} \in \operatorname{Ker}(d_p)$ defines a class in $H^p(X,F)$. It is straightforward to verify that this class depends only on α and not on our choices of $c_0,\ldots,c_p,c_{p+1}$. Our construction therefore defines the required homomorphism $p(\mathcal{U})$.

Property A of the Proposition follows immediately since the sequence $c_0,\ldots,c_{p+1}$ is mapped by a into a sequence $c_0,\ldots,c_{p+1}$ defining $p(\mathcal{U})(a(\alpha))$.

If the conditions of B hold then rows, as well as columns, of our diagram are exact (excluding initial row and column) and so, by symmetry, we may construct an inverse for $p(\mathcal{U})$. □

Remarks.

1. We call a cover satisfying the conditions of 1B a *Leray cover* (for F). Notice that the existence of a Leray cover for F may imply that higher dimensional sheaf cohomology groups vanish. Thus, if $U_s = \emptyset$, $s \in I^{p+1}$, $p \geq p_0$, we see that $H^p(X,F) = 0$, $p \geq p_0$.

2. It is clearly important to find Leray covers for a given sheaf F. For example one can show, using a Riemannian metric, that every differential manifold has a cover by convex open sets (Helgason [1; page 54]). Since intersections of convex sets are convex and convex sets are contractible, it follows that differential manifolds admit Leray covers for the constant sheaves $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, etc. At a much deeper level, we shall show later that Stein open covers of a complex manifold are Leray covers for an important class of $\mathcal{O}$-modules.

3. The proof of Theorem 6.3.16 clearly works for any fine resolution of F - not just the canonical resolution. We frequently use this observation in the sequel.

Example 9. Let $\mathcal{U} = \{U_i\}$ be an open cover of the differential manifold X. We give an explicit computation for the map $p(\mathcal{U})\colon H^p(\mathcal{U},\mathbb{C}) \to H^p_{DR}(X,\mathbb{C})$ in case $p = 2$. First choose a C^∞ partition of unity $\{\theta_i\}$ of X subordinate to the cover $\mathcal{U}$. Let $\{c_{ijk}\} \in Z^2(\mathcal{U},\mathbb{C})$ respresent the class $\alpha \in H^2(\mathcal{U},\mathbb{C})$. Define $\{\phi_{jk}\} \in C^1(\mathcal{U},\underline{C}^0)$ by

$$\phi_{jk} = \sum_i \theta_i c_{ijk}.$$

Certainly we have $D_0(\{\phi_{jk}\}) = d_{-1}\{c_{ijk}\}$ $(= \{c_{ijk}\})$. Again define $\{\phi_k\} \in C^0(\mathcal{U},\underline{C}^1)$ by

$$\phi_k = \sum_j \theta_j d\phi_{jk}\ .$$

We have $D_1(\{\phi_j\}) = d\{\phi_{jk}\}$. Finally observe that $d\phi_k = d\phi_l$ on U_{kl} and so $\gamma = \{d\phi_k\}$ is a well defined closed 2-form on X which representa the class $p(\mathcal{U})(\alpha)$ in $H^2_{DR}(X,\mathbb{C})$. The construction for $p \neq 2$ is similar.

Proposition 6.3.17. Let $\mathcal{U}$, $\mathcal{V}$ be open covers of X and $\mathcal{V}$ be a refinement of $\mathcal{U}$. Given a sheaf $\mathcal{F}$ on X there exists a canonical homomorphism

$$p(\mathcal{U},\mathcal{V}): H^p(\mathcal{U},\mathcal{F}) \to H^p(\mathcal{V},\mathcal{F})$$

satisfying

1. The diagram

$$\begin{array}{ccc} H^p(\mathcal{U},\mathcal{F}) & \xrightarrow{p(\mathcal{U},\mathcal{V})} & H^p(\mathcal{V},\mathcal{F}) \\ & {\scriptstyle p(\mathcal{U})} \searrow \quad \swarrow {\scriptstyle p(\mathcal{V})} & \\ & H^p(X,\mathcal{F}) & \end{array}$$

commutes, $p \geq 0$.

2. If $a:\mathcal{F} \to \mathcal{G}$ is a morphism, we have

$$p(\mathcal{U},\mathcal{V})a^p_{\mathcal{U}} = a^p_{\mathcal{V}}p(\mathcal{U},\mathcal{V}), \quad p \geq 0 .$$

3. If $\mathcal{W}$ is a refinement of $\mathcal{V}$, we have $p(\mathcal{U},\mathcal{W}) = p(\mathcal{V},\mathcal{W})p(\mathcal{U},\mathcal{V})$.

Proof. Let $\mathcal{V} = \{V_j: j \in J\}$, $\mathcal{U} = \{U_i: i \in I\}$ and fix a *refinement* mapping $\phi: J \to I$ satisfying $V_j \subset U_{\phi(j)}$, $j \in J$. Given $c \in C^p(\mathcal{U},\mathcal{F})$, define $\tilde{\phi}c \in C^p(\mathcal{V},\mathcal{F})$ by

$$(\phi c)_s = c_{\phi(s_0)\ldots\phi(s_p)}|V_s, \quad s = (s_0,\ldots,s_p) \in I^{p+1}.$$

Since $\tilde{\phi}$ obviously commutes with the coboundary operators D, $\tilde{\phi}$ induces a map $\phi_*: H^p(\mathcal{U},\mathcal{F}) \to H^p(\mathcal{V},\mathcal{F})$. We claim that ϕ_* is independent of the choice of ϕ. Suppose that ϕ' is another refinement mapping. We construct a "homotopy" operator $H: C^{p+1}(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{V},\mathcal{F})$ between $\tilde{\phi}$ and $\tilde{\phi}$. To do this we first give J a total ordering. Then if $s_0 < \ldots < s_p$ and $c \in C^{p+1}(\mathcal{U},\mathcal{F})$ we define

$$(Hc)_s = \sum_{j=0}^{p} (-1)^j c_{\phi(s_0)\ldots\phi(s_j)\phi'(s_j)\ldots\phi'(s_p)} \cdot$$

Extend H to all cochains by requiring that H is alternating in s. Computing we find that

$$HD + DH = \tilde{\phi}' - \tilde{\phi} .$$

Hence, if $c \in Z^p(U,F)$, we have $(\tilde{\phi}' - \tilde{\phi})(c) \in B^p(V,F)$. Therefore $\phi_* = \phi'_*$ and we may set $p(U,V) = \phi_*$ for any choice of refinement mapping ϕ. The proofs of the remaining statements of the proposition are elementary and we leave details to the reader. □

An immediate corollary of Proposition 6.3.17 is that $\{H^p(U,F), p(U,V)\}$ constitutes a direct system. Hence we may define

$$\check{H}^p(X,F) = \mathrm{dirlim}_U H^p(U,F).$$

We call $\check{H}^p(X,F)$ the *pth. Čech cohomology group of X with coefficients in* F.

By part 1 of Proposition 6.3.17, we see that for $p \geq 0$ we have canonical homomorphisms

$$\chi(U): H^p(U,F) \to \check{H}^p(X,F)$$

$$\chi: \check{H}^p(X,F) \to H^p(X,F) .$$

These homomorphisms satisfy the usual naturality properties. For example, if $a: F \to G$ is a morphism of sheaves we have induced homomorphisms $\check{a}^p: \check{H}^p(X,F) \to \check{H}^p(X,G)$ satisfying the conditions of part 2 of Theorem 6.3.10 and

$$\chi\check{a}^p = \chi a^p , \ p \geq 0.$$

Theorem 6.3.18. For paracompact spaces, Čech and sheaf cohomology groups are isomorphic.

To prove this result, it is sufficient to show that the Čech groups satisfy all the properties of Theorem 5.3.10. In view of what we have proved already, it remains to construct a long exact cohomology sequence for $\check{H}$. Actually, we shall prove a little more. We shall show that Čech and sheaf cohomology theory are canonically isomorphic with canonical isomorphisms being given by the maps χ. In particular, we shall show that the maps χ commute with connecting homomorphisms.

Theorem 6.3.19. Let $0 \to F \xrightarrow{a} G \xrightarrow{b} H \to 0$ be a short exact sequence of sheaves on X. For $p \geq 0$ we have a connecting homomorphism $\delta\colon \check{H}^p(X,H) \to \check{H}^{p+1}(X,F)$ satisfying

A. The Čech cohomology sequence

$$0 \to \check{H}^0(X,F) \xrightarrow{\check{a}^0} \check{H}^0(X,G) \xrightarrow{\check{b}^0} \check{H}^0(X,H) \xrightarrow{\delta} \check{H}^1(X,F) \xrightarrow{\check{a}^1} \cdots$$

is exact.

B. The diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \check{H}^0(X,F) & \xrightarrow{\check{a}^0} & \check{H}^0(X,G) & \xrightarrow{\check{b}^0} & \check{H}^0(X,H) & \xrightarrow{\delta} & \check{H}^1(X,F) & \xrightarrow{\check{a}^1} & \cdots \\ & & \downarrow\chi & & \downarrow\chi & & \downarrow\chi & & \downarrow\chi & & \\ 0 & \longrightarrow & H^0(X,F) & \xrightarrow{a^0} & H^0(X,G) & \xrightarrow{b^0} & H^0(X,H) & \xrightarrow{\sigma} & H^1(X,F) & \xrightarrow{a^1} & \cdots \end{array}$$

commutes.

Proof. Let U be an open cover of X and $p \geq 0$. Set $\tilde{C}^p(U,H) = bC^p(U,G)$. Then the sequence

$$0 \to C^p(U,F) \to C^p(U,G) \to \tilde{C}^p(U,H) \to 0$$

is exact. Since $D\tilde{C}^{p-1}(U,H) \subset \tilde{C}^p(U,H)$ we may define the "cohomology" group $\tilde{H}^p(U,H)$ to be $\tilde{Z}^p(U,H)/\tilde{B}^p(U,H)$. Exactly as in the proof of 3 in Theorem 6.3.10 we may define a connecting homomorphism $\tilde{\delta}\colon \tilde{H}^p(U,H) \to H^{p+1}(U,F)$ and obtain a long exact sequence for the cohomology groups $H^*(U,F)$, $H^*(U,G)$, $\tilde{H}^*(U,H)$. Everything commutes with direct limits and so we arrive at the long exact cohomology sequence

$$\cdots \to \tilde{H}^p(X,H) \longrightarrow \check{H}^{p+1}(X,F) \xrightarrow{\check{a}^{p+1}} \check{H}^{p+1}(X,G) \xrightarrow{\check{b}^{p+1}} \tilde{H}^{p+1}(X,H) \to \cdots$$

We claim that $\tilde{H}^p(X,H) \approx \check{H}^p(X,H)$. For this it suffices to show that if $h \in C^p(U,H)$, we can find a refinement $V = \{V_j: j \in J\}$ of $U = \{U_i: i \in I\}$ and refinement map $\phi: J \to I$ such that $\tilde{\phi}h \in \tilde{C}^p(V,H)$. Since X is paracompact, we may assume that U is a locally finite cover of X. Let $W = \{W_i\}$ be a refinement of U such that $\bar{W}_i \subset U_i$, $i \in I$. Choose an open neighbourhood V_x of each point $x \in X$ such that

a) If $x \in W_i$, $V_x \subset W_i$ and if $V_x \cap W_j \neq \emptyset$, then $V_x \subset U_j$.

b) If $x \in U_s$, $s \in I^{p+1}$, then $V_x \subset U_s$.

Observe that a) and b) continue to hold for open neighbourhoods of x contained in V_x. Since $b: G \to H$ is surjective we may, shrinking V_x if necessary, find $g_s^x \in G(V_x)$ such that $bg_s^x = h_s|V_x$. Choose a refining map $\phi: X \to I$ for the covers $V = \{V_x: x \in X\}$ and U satisfying $x \in W_{\phi(x)}$ for all $x \in X$. Let $t = (t_0,\ldots,t_p) \in X^{p+1}$ and suppose $V_t \neq \emptyset$. By choice of ϕ, $V_{t_0} \cap W_{\phi(t_j)} \neq \emptyset$, $0 \leq j \leq p$, and so by a) $V_{t_0} \subset U_{\phi(t_j)}$, $0 \leq j \leq p$. Hence $V_{t_0} \subset U_{\phi(t)}$, where we have used the abbreviated notation $\phi(t)$ for $\phi(t_0)\ldots\phi(t_p)$. For $t \in X^{p+1}$, we define $g_t \in G(V_t)$ by

$$g_t = g_t^{t_0} .$$

Clearly $bg_t = h_{\phi(t)}|V_t$. Hence $\tilde{\phi}h \in \tilde{C}^p(V,H)$.

All that remains to be proved is statement B of the theorem. For this we take canonical resolutions of F, G and H, fix an open cover U of X and take appropriate complexes of cochains over each term in the canonical resolutions as in the proof of Theorem 6.3.13. We obtain a 3-dimensional commutative diagram of sequences. It is then a straightforward diagram chase to verify that the definition of $\tilde{\delta}: \tilde{H}^p(U,H) \to H^{p+1}(U,F)$ is compatible with that of $\delta: H^p(X,H) \to H^{p+1}(X,F)$. Essentially we have to check that the maps $p(U)$ map the defining sequences for $\tilde{\delta}$ down to defining sequences for δ. We omit the lengthy details. Finally take direct limits. □

Remarks.

1. An alternative construction of the long exact sequence of Čech cohomology, based on sheaves of cochains (cf. Example 3) may be found in Godement [1].

2. The paracompactness assumption implicit in Theorem 6.3.19 is essential: Čech cohomology need not be exact for non-paracompact spaces. However, Leray's theorem continues to hold and this fact was exploited by Serre in his foundational paper on coherent sheaves in algebraic geometry (Serre [2]).

We end this chapter with some important examples and computations.

Examples.

10. Let X be a differential manifold with structure sheaf $\mathcal{D}$ and let $\mathcal{D}^*$ denote the (multiplicative) sheaf of groups of units of $\mathcal{D}$. We claim that the group CLB(X) of isomorphism classes of complex line bundles on X is canonically isomorphic to $H^1(X,\mathcal{D}^*)$. Let $\xi \in H^1(X,\mathcal{D}^*)$. Now $H^1(X,\mathcal{D}^*) \approx \check{H}^1(X,\mathcal{D}^*)$ and so we may find an open cover $\mathcal{U} = \{U_i\}$ of X and $\{\phi_{ij}\} \in Z^2(\mathcal{U},\mathcal{D}^*)$ such that $\chi(\mathcal{U})$ maps the cohomology class of $\{\phi_{ij}\}$ to ξ. The cocycle conditions on $\{\phi_{ij}\}$ imply that $\phi_{ij}\phi_{jk} = \phi_{ik}$, $i,j,k \in I$. Since $\phi_{ij}\colon U_{ij} \to \mathbb{C}^{\bullet} \approx GL(1,\mathbb{C})$, we see that the ϕ_{ij} are the transition functions for a complex line bundle $L(\xi)$ on X. We leave it to the reader to check that $L(\xi)$ depends only on ξ and not on our particular choice of cover or cocycle and that the map $\xi \to L(\xi)$ is an isomorphism of $H^1(X,\mathcal{D}^*)$ with CLB(X).

We may use this isomorphism between CLB(X) and $H^1(X,\mathcal{D}^*)$ to define an important topological invariant of complex line bundles. First observe that we have an exact sheaf sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{D} \xrightarrow{\exp} \mathcal{D}^* \to 0 .$$

Here $i\colon \mathbb{Z} \to \mathcal{D}$ denotes the inclusion and $\exp\colon \mathcal{D} \to \mathcal{D}^*$ is defined by $\exp_U(f)(x) = \exp(2\pi i f(x))$, $x \in U$, $f \in \mathcal{D}(U)$. The surjectivity of exp follows by noting that $\exp_U$ has inverse $(2\pi i)^{-1}\log$ on simply connected open subsets U.

Consider the following portion of the long exact cohomology sequence

$$\ldots \to H^1(X,\mathcal{D}) \to H^1(X,\mathcal{D}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X,\mathcal{D}) \to \ldots$$

Since $\mathcal{D}$ is a fine sheaf, $H^1(X,\mathcal{D}) = H^2(X,\mathcal{D}) = 0$ and so $\delta: H^1(X,\mathcal{D}^*) \approx H^2(X, \mathbb{Z})$. Suppose $L \in H^1(X,\mathcal{D}^*)$ is a complex line bundle. Define $c_1(L) = -\delta(L) \in H^2(X, \mathbb{Z})$. We call $c_1(L)$ the *first Chern class* of L. We see that $L = \underline{\mathbb{C}}$ - the trivial line bundle - if and only if $c_1(L) = 0$. Fix an open cover $\mathcal{U} = \{U_i: i \in I\}$ of X such that all intersections U_s, $s \in I^{p+1}$, $p \geq 0$, are contractible. For example, we can choose a cover of X by contractible open sets (see Remark 2 following Theorem 6.3.16). Applying Leray's theorem, we can see that $H^2(X, \mathbb{Z}) \approx H^2(\mathcal{U}, \mathbb{Z})$ and $H^1(X,\mathcal{D}^*) \approx H^1(\mathcal{U},\mathcal{D}^*)$. Given $L \in CLB(X)$ we may therefore find transition functions $\{\phi_{ij}\}$ for L defined relative to the cover $\mathcal{U}$. Every intersection U_{ij} is simply connected and so we may choose a branch of $\log\phi_{ij}$ on each U_{ij}. The cocycle conditions on $\{\phi_{ij}\}$ imply that for all $i,j,k \in I$,

$$c_{ijk} = (2\pi i)^{-1}(\log\phi_{ij} + \log\phi_{jk} + \log\phi_{ki}) \in \mathbb{Z},$$

and so $\{c_{ijk}\} \in Z^2(\mathcal{U},\mathbb{Z})$. By the construction of the connecting homomorphism, it follows that $\{c_{ijk}\}$ is a representative for $c_1(L) \in H^2(X, \mathbb{Z})$.

11. Let X be a complex manifold. As in Example 10 we have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

of sheaves on X and we see that $H^1(X,\mathcal{O}^*)$ is isomorphic to the group $HLB(X)$ of holomorphic line bundles on X. We have the commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H^1(X,\mathcal{O}) & \longrightarrow & H^1(X,\mathcal{O}^*) & \xrightarrow{\delta} & H^2(X, \mathbb{Z}) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \| & & \\
\ldots & \longrightarrow & H^1(X,\mathcal{D}) & \longrightarrow & H^1(X,\mathcal{D}^*) & \xrightarrow{\delta} & H^2(X, \mathbb{Z}) & \longrightarrow & \ldots
\end{array}$$

Here the vertical maps are induced by the natural inclusions of $\mathcal{O}$ in $\mathcal{D}$, $\mathcal{O}^*$ in $\mathcal{D}^*$ and the identity map of $\mathbb{Z}$. Hence, as in Example 10, the map δ: $H^1(X,\mathcal{O}^*) \to H^2(X, \mathbb{Z})$ is minus the first Chern class map. However, the map will generally not be an isomorphism as neither of $H^1(X,\mathcal{O})$ and $H^2(X,\mathcal{O})$ need vanish. This is just a reflection of the fact that a holomorphic line bundle may be trivial in CLB(X) but not trivial in HLB(X). The existence of a Leray cover for $\mathcal{O}^*$ is now a non-trivial matter as it depends on being able to find covers which are Leray for $\mathcal{O}$ as well as $\mathbb{Z}$.

12. We continue with the notation and assumptions of Example 11. Let $j: \mathbb{Z} \to \mathbb{C}$ denote the inclusion map of constant sheaves. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{HLB}(X) \approx H^1(X,\mathcal{O}^*) & \xrightarrow{\ \delta\ } & H^2(X, \mathbb{Z}) \\ & \searrow^{\theta} & \downarrow j \\ & & H^2(X,\mathbb{C}) \\ & & \wr\wr \\ & & H^2_{DR}(X,\mathbb{C}) \ . \end{array}$$

Here θ: $H^1(X,\mathcal{O}^*) \to H^2_{DR}(X,\mathbb{C})$ is defined to be the composition of δ with j and the canonical isomorphism between $H^2(X,\mathbb{C})$ and $H^2_{DR}(X,\mathbb{C})$. We let $c_1(L)_{\mathbb{C}} = -\theta(L) \in H^2_{DR}(X,\mathbb{C})$, $L \in \mathrm{HLB}(X)$. We are going to give an explicit representation of $c_1(L)_{\mathbb{C}}$ as a closed 2-form on X. Suppose that $\{\phi_{ij}\}$ are the transition functions of $L \in \mathrm{HLB}(X)$ relative to some open cover $\mathcal{U} = \{U_i\}$ of X. As in Example 10 we may and shall assume that all intersections U_s are contractible. As we showed in Example 10, $c_1(L) \in H^2(X, \mathbb{Z})$ is represented by the cocycle

$$\{c_{ijk}\} = \{(2\pi i)^{-1}(\log\phi_{ij} + \log\phi_{jk} + \log\phi_{ki})\} \in Z^2(\mathcal{U},\mathbb{Z}).$$

Let us assume for the moment that there exist $\{a_i\} \in C^0(\mathcal{U},\mathcal{D}^*)$ such that

$$|\phi_{ij}|^2 = a_j/a_i \ .$$

(This amounts to claiming that $L^* \otimes \bar{L}^* = \underline{\mathbb{C}}$ in CLB(X)). As in the proof of Theorem 6.3.16 (see also Example 9), we first construct $\{f_i\} \in C^0(\mathcal{U},\underline{C}^1)$ such that

$$\frac{1}{2\pi i}\, d\log\phi_{ij} = f_j - f_i .$$

For this we may take $f_i = \frac{1}{2\pi i}\, \partial\log a_i$, since

$$\log\phi_{ij} + \overline{\log\phi_{ij}} = \log a_j - \log a_i$$

and so

$$d\log\phi_{ij} = \partial\log\phi_{ij} = \partial\log a_j - \partial\log a_i .$$

Our required 2-form representing $c_1(L)_{\mathbb{C}}$ is then given by

$$\begin{aligned}\psi &= -\{df_i\} \\ &= -\{\tfrac{1}{2\pi i}\, \bar\partial\partial\log a_i\} \\ &= \{-\tfrac{i}{2\pi}\, \partial\bar\partial\log a_i\}.\end{aligned}$$

Notice that $c_1(L)_{\mathbb{C}}$ is represented by a (1,1)-form. In Example 15 we shall show that every "integral" (1,1)-form is the Chern class of some holomorphic line bundle. The reader should also observe that we do not really need to assume that the domains U_{ij} are simply connected as the indeterminancy in $\log\phi_{ij}$ drops out when we take exterior derivations.

Finally, let us justify our assumption that $L^* \otimes \bar{L}^*$ is isomorphic to the trivial line bundle. First observe that $L^*_z \otimes \bar{L}^*_z$ is the space of Hermitian forms on L_z, $z \in X$. Now any convex combination of positive definite Hermitian quadratic forms is positive definite. Therefore, taking trivialisations of L over $\mathcal{U}$, we may choose a smooth section of $L^* \otimes \bar{L}^*$ over each U_i which defines a positive definite Hermitian form on the fibres. Glueing together using a smooth partition of unity we obtain a globsl section of $L^* \otimes \bar{L}^*$ which restricts to a positive definite Hermitian form on the fibres. Hence we have obtained a non-vanishing smooth section of $L^* \otimes \bar{L}^*$.

13. We compute the 1st. Chern class of the hyperplane section bundle H of $P^1(\mathbb{C})$. Let $\mathcal{U} = \{U_0, U_1\}$ be the open cover of $P^1(\mathbb{C})$ corresponding to the canonical atlas. The transition functions for

the hyperplane section bundle are given by

$$\phi_{01}(z_0, z_1) = z_1/z_0$$

and setting

$$a_0(z_0, z_1) = |z_0|^2/(|z_0|^2 + |z_1|^2)$$

$$a_1(z_0, z_1) = |z_1|^2/(|z_0|^2 + |z_1|^2)$$

we see that $|\phi_{01}|^2 = a_1/a_0$. Hence our representative μ for $c_1(H)_{\mathbb{C}}$ is given by $\mu = \{\mu_i\} = \{-\frac{i}{2\pi}\partial\bar{\partial}\log a_i\}$. Setting $z_0/z_1 = t$, we see that

$$\begin{aligned}\mu_1(t) &= -\frac{i}{2\pi}\partial\bar{\partial}\log(1/(1+|t|^2)) \\ &= \frac{i}{2\pi}(1+|t|^2)^{-2}dtd\bar{t}\ .\end{aligned}$$

Now for compact Riemann surfaces, integration defines a canonical isomorphism $H^2(X,\mathbb{C}) \approx \mathbb{C}$ and so we may regard $c_1(H)_{\mathbb{C}}$ as lying in $\mathbb{C}$. Thus

$$\begin{aligned}c_1(H)_{\mathbb{C}} = \int_{P^1(\mathbb{C})}\mu &= \int_{\mathbb{C}}\mu_1 \\ &= \frac{i}{2\pi}\int_{\mathbb{C}}(1+|t|^2)^{-2}dtd\bar{t} \\ &= \frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi}(1+r^2)^{-2}rdrd\theta \\ &= +1.\end{aligned}$$

Since $H^2(X, \mathbb{Z}) \approx \mathbb{Z}$ for compact Riemann surfaces, we have proved $c_1(H) = +1$. Our choice of sign for c_1 was made precisely so that the hyperplane section bundle of $P^1(\mathbb{C})$ had positive Chern class...

14. Let L be a holomorphic line bundle on the compact Riemann surface L and suppose that $M^*(L) \neq \emptyset$. Recall (proposition 1.5.7) that if $s \in M^*(L)$ then deg(div(s)) is independent of s and that the degree of L, deg(L), is defined to be deg(div(s)). Using the canonical isomorphism of $H^2(M, \mathbb{Z})$ with $\mathbb{Z}$ we may regard $c_1(L)$ as lying in $\mathbb{Z}$. We claim that $c_1(L) = \deg(L)$. Fix $s \in M^*(L)$ and let

$\operatorname{div}(s) = \sum_{i=1}^{n} n_i \cdot z_i$. Choose a finite cover $\{U_i: i = 1,\ldots,m\}$ of M satisfying

a) $L|U_i$ is holomorphically trivial.

b) $z_i \in U_i$, $1 \le i \le n$.

c) There exists an open neighbourhood V_i of each z_i such that V_i is biholomorphic to the unit disc in $\mathbb{C}$ and $V_i \cap \bigcup_{j \ne i} U_j = \emptyset$.

Denote the corresponding set of transition functions for L by $\{\phi_{jk}: U_{jk} \to \mathbb{C}^*\}$. Let s_j denote the local representative of s on U_j. Then s_j, s_k will be holomorphic on U_{jk}, $j \ne k$, and

$$|\phi_{jk}|^2 |s_k|^2 = |s_j|^2 \text{ on } U_{jk} .$$

For $i > n$, set $g_i = |s_i|^2$. For $i \le n$, we may, using bump functions, modify $|s_i|^2$ to obtain a C^∞ positive function g_i on U_i satisfying

$$g_i | U_i \setminus V_i = |s_i|^2 .$$

In particular, $g_i | \partial V_i = |s_i|^2$ and, by condition c) above, we will have

$$|\phi_{jk}|^2 g_k = g_j \text{ on } U_{jk} \text{ for all } j,k .$$

Hence, by Example 12, we have

$$\begin{aligned} c_1(L)_{\mathbb{C}} = c_1(L) &= -\frac{i}{2\pi} \int_M \partial\bar\partial \log g_j^{-1} \\ &= \frac{1}{2\pi i} \int_M \bar\partial\partial \log g_j \\ c_1(L) &= \frac{1}{2\pi i} \sum_{i=1}^{n} \int_{V_i} \bar\partial\partial \log g_i . \end{aligned}$$

Now $d\partial \log g_i = \bar\partial\partial \log g_i$ and so, by Stokes' theorem,

$$\begin{aligned} c_1(L) &= \frac{1}{2\pi i} \sum_{i=1}^{n} \int_{\partial V_i} \partial \log g_i \\ &= \frac{1}{2\pi i} \sum_{i=1}^{n} \int_{\partial V_i} d \log s_i \end{aligned}$$

$$= \sum_{i=1}^{n} n_i, \text{ by the residue theorem.}$$

$$= \deg(L).$$

15. We showed in Example 12 that the Chern class of a holomorphic line bundle on X could be represented, via the de Rham isomorphism, by a closed (1,1)-form on X. We shall now show that every integral closed (1,1)-form is the Chern class of some holomorphic line bundle on X (modulo torsion). Let $j\colon \mathbb{Z} \to \mathbb{C}$ denote the inclusion map of constant sheaves and $H^{1,1}_{DR}(X,\mathbb{C})$ denote the subgroup of $H^2_{DR}(X,\mathbb{C})$ admitting representatives by closed (1,1)-forms. Set $H^{1,1}(X,\mathbb{Z}) = H^{1,1}_{DR}(X,\mathbb{C}) \cap jH^2(X,\mathbb{Z})$ We claim that $c_1(H^1(X,\mathcal{O}^*))_{\mathbb{C}} = H^{1,1}(X,\mathbb{Z})$. Let $i\colon \mathbb{Z} \to \mathcal{O}$, $k\colon \mathbb{C} \to \mathcal{O}$ denote the inclusion maps of the constant sheaves $\mathbb{Z}$, $\mathbb{C}$ in $\mathcal{O}$. We have the commutative diagram

$$\begin{array}{ccccc} H^1(X,\mathcal{O}^*) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) & \xrightarrow{i} & H^2(X,\mathcal{O}) \\ & & & \searrow^{j} \quad \swarrow_{k} & \\ & & & H^2(X,\mathbb{C}) & . \end{array}$$

Since we already know that $c_1(H^1(X,\mathcal{O}^*))_{\mathbb{C}} \subset H^{1,1}(X,\mathbb{Z})$, it is enough to prove that $kH^{1,1}_{DR}(X,\mathbb{C}) = 0$. Taking the de Rham and Dolbeault resolutions of $\mathbb{C}$ and $\mathcal{O}$ we have the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \underline{C}^0 & \xrightarrow{d} & \underline{C}^1 & \xrightarrow{d} & \underline{C}^2 & \xrightarrow{d} & \cdots \\ & & \downarrow k & & \downarrow I & & \downarrow Q_1 & & \downarrow Q_2 & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \underline{C}^{0,0} & \xrightarrow{\bar\partial} & C^{0,1} & \xrightarrow{\bar\partial} & \underline{C}^{0,2} & \xrightarrow{\bar\partial} & \cdots \end{array}$$

where the morphism $Q_p\colon \underline{C}^p \to \underline{C}^{0,p}$ is just projection of p-forms onto the (0,p)-component. The map $k^p\colon H^p(X,\mathbb{C}) \to H^p(X,\mathcal{O})$ is therefore represented by

$$\begin{array}{ccc} Q_p\colon \mathrm{Ker}(d) & \longrightarrow & \mathrm{Ker}(\bar\partial) \\ \cap & & \cap \\ C^p(M) & & C^{0,p}(M) \end{array}$$

Since Q_2 maps (1,1)-forms to zero it follows that the image of $H^{1,1}_{DR}(X,\mathbb{C})$ in $H^2(X,\mathcal{O})$ is zero.

16. Associated to the divisor sequence $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$ on a complex manifold X we have the long exact sequence

$$\begin{array}{ccccccccc} 0 \to & H^0(X,\mathcal{O}^*) & \to & H^0(X,\mathcal{M}^*) & \to & H^0(X,\mathcal{D}) & \xrightarrow{\delta} & H^1(X,\mathcal{O}^*) & \to \ \cdots \\ & \| & & \| & & \| & & \| & \\ & A^*(X) & & \mathcal{M}^*(X) & & \mathcal{D}(X) & & HLB(X) & \end{array}$$

Recall from §9, Chapter 5, that we have a homomorphism

$$[\ ]: \mathcal{D}(X) \to HLB(X).$$

By the definition of δ in Čech theory it is clear that

$$\delta(d) = -[d],\ d \in \mathcal{D}(X).$$

So we see again that a divisor d is the divisor of a meromorphic function if and only if [d] is a holomorphically trivial line bundle. Recalling from Example 12 that the Chern class map c_1: $HLB(X) \to H^2(X,\mathbb{Z})$ is defined to be *minus* the connecting homomorphism δ: $HLB(X) \to H^2(X,\mathbb{Z})$, we see that $c_1([d]) = \delta\delta(d)$. In summary, we see that for a divisor d to be the divisor of a meromorphic function there are precisely two obstructions. Firstly a topological obstruction: $c_1([d])$ must be zero; secondly an analytic obstruction: [d] must be holomorphically trivial.

For the remainder of this example, we shall assume that X is projective algebraic. We shall show later that every holomorphic line bundle on X admits a non-trivial meromorphic section. This clearly implies that the image of $H^1(X,\mathcal{O}^*)$ in $H^1(X,\mathcal{M}^*)$ is zero and so we obtain the exact sequence

$$0 \to A^*(X) \to \mathcal{M}^*(X) \to \mathcal{D}(X) \xrightarrow{\delta} HLB(X) \to 0 .$$

Hence, $HLB(X) \cong \mathcal{D}(X)/\mathcal{L}(X)$, where $\mathcal{L}(X)$ is the group of linear equivalence classes of divisors on X.

From Example 14, we see that given $\eta \in H^{1,1}(X,\mathbb{Z})$, there exists $d \in \mathcal{D}(X)$ such that $c_1([d])_{\mathbb{C}} = \eta$. It may be shown (see Griffiths and Harris [1]) that d defines a class in $H_{2n-2}(X,\mathbb{Z})$ which is the Poincaré dual of η with respect to the pairing $H_{2n-2}(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^{2n}(X,\mathbb{Z}) \approx \mathbb{Z}$. So we have a special case of the *Hodge conjecture* due to Lefschetz: Every class in $H^{1,1}(X,\mathbb{Z})$ has a Poincaré dual represented by an integral combination of analytic hypersurfaces in X (modulo torsion).

Finally let Pic(X) denote the subgroup of HLB(X) consisting of holomorphic line bundles which are trivial as complex line bundles. Thus $\mathrm{Pic}(X) = \mathrm{Ker}(c_1)$. Since X is compact it is easily seen from the cohomology sequence of $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$ that $\mathrm{Pic}(X) \cong H^1(X,\mathcal{O})/H^1(X,\mathbb{Z})$. Furthermore it can be shown that Pic(X) has the natural structure of a compact connected complex Lie group. That is, a complex torus. Pic(X) is called the *Picard variety* of X. The Picard variety is an important birational invariant of X. It is an Abelian variety (for this it is sufficient that X be Moishezon). Proofs of these statements, together with further references, may be found in Ueno [1] and Griffiths and Harris [1].

Exercises.

1. Verify that
   a) $H^p(D,\mathcal{O}) = 0$, $p \geq 1$, for every domain D in $\mathbb{C}$.
   b) $H^p(D,\Omega^q) = 0$, $p \geq 1$, $q \geq 0$, for every open polydisc D in $\mathbb{C}^n$.

2. Let G be a group, not necessarily abelian. Show how to define $\check{H}^1(X,G)$ and prove that $\check{H}^1(X,GL(n,\mathbb{C}))$ is isomorphic to the set of isomorphism classes of n-dimensional complex vector bundles on X.

3. Prove that the map $\rho(\mathcal{U})$: $H^1(\mathcal{U},F) \to H^1(X,F)$ is injective for all open covers $\mathcal{U}$ of X.

4. Verify that the de Rham cohomology groups of a differential manifold are topological invariants (as opposed to differential topological invariants) without using the isomorphism between singular and de Rham groups.

5. Let X be a Riemann surface. Verify that the sheaves $\mathcal{D}$ and $\mathcal{M}/\mathcal{D}$ are soft. Is this result true for general complex manifolds?

6. Show that $\mathrm{Pic}(X) = \{\mathbb{C}\}$ in case X is any domain in $\mathbb{C}$ or an open polydisc in $\mathbb{C}^n$.

7. Prove that $c_1(TP^1(\mathbb{C})) = +2$.

8. $H^2(P^n(\mathbb{C}),\mathbb{Z}) \cong \mathbb{Z}$. Let H denote the hyperplane section bundle of $P^n(\mathbb{C})$. Prove that $c_1(H)$ generates $H^2(P^n(\mathbb{C}),\mathbb{Z})$.

9. Let $f: X \to Y$ be holomorphic and $L \in HLB(Y)$. Prove that $f^*(c_1(L)_{\mathbb{C}}) = c_1(f^*L)_{\mathbb{C}}$.

10. Let $T = \mathbb{C}^n/\Lambda$ be an n-dimensional complex torus with period lattice Λ and suppose that $L(H,m)$ is a holomorphic line bundle on T (for notation, see §9, Chapter 5 and recall that H is an Hermitian form on $\mathbb{C}^n$ whose imaginary part is integral on $\Lambda \times \Lambda$). Prove

a) The function $a(z,t) = \exp(-\pi H(z,z))|t|^2$ on $\mathbb{C}^n \times \mathbb{C}$ is invariant under the action
$(z,t) \to (z+\lambda, \exp(-\pi(2\mathrm{Re}(H(z,\lambda)) + H(\lambda,\lambda)))|t|^2)$, $\lambda \in \Lambda$.
Deduce that a induces a smooth map $\tilde{a}: L(H,m) \to \mathbb{R}$ which is quadratic on fibres (see also Exercise 6, §9, Chapter 5).

b) The first Chern class of $L(H,m)$ is represented by the form $-\frac{1}{2\pi}\partial\bar{\partial}\log\exp(-\pi H(z,z)) = \frac{i}{2}H$.

c) Every 1-dimensional complex torus admits a holomorphic line bundle of Chern class +1 (Integrate $\frac{i}{2}H$ over a period parallelogram).

11. Recall that a sheaf $\mathcal{F}$ of groups over a topological space X (not necessarily paracompact) is *flabby* if for all open subsets U of X the sequence $\mathcal{F}(X) \to \mathcal{F}(U) \to 0$ is exact. Prove

a) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of sheaves over X and $\mathcal{F}$ is flabby then the sequence is presheaf exact. If, in addition $\mathcal{G}$ is flabby, deduce that $\mathcal{H}$ is flabby.

b) If X is paracompact, then $\mathcal{F}$ flabby implies $\mathcal{F}$ soft.

12. Let X be a closed subset of the topological space Z and F be a sheaf of groups on X. As in exercise 4, §1, let $\tilde{F}$ denote the trivial extension of F to Z. Prove that for $p \geq 0$, $H^p(Z,\tilde{F}) \approx H^p(X,F)$. (Look at the trivial extension of the canonical resolution of F).

13*. Let $f: X \to Y$ be a holomorphic map between complex manifolds and F be a sheaf of O_X-modules on X. For $p \geq 0$, let $R^p f_* F$ be the sheaf of O_Y-modules on Y associated to the presehaf $U \to H^p(f^{-1}(U),F)$. Show that for any exact sequence $0 \to F \to G \to H \to 0$ of O_X-modules on X, the induced sequence

$$0 \to f_*F \to f_*G \to f_*H \to R^1f_*F \to R^1f_*G \to \ldots$$

is exact.

14. Let $f: X \to Y$ be a homeomorphism between topological spaces X and Y, F and G be sheaves on X and Y respectively and $a: G \to F$ be a morphism of sheaves covering f^{-1}. Show that for $p \geq 0$, we have naturally induced homomorphisms

$$f(a)^*: H^p(Y,G) \to H^p(X,F)$$

(Use Čech theory).

15. Let $f: X \to Y$ be s continuous map between topological spaces X, Y and F be a sheaf on Y. Show that there exist homomorphisms $f^*: H^p(Y,F) \to H^p(X,f^{-1}F)$ satisfying

a) If $f = \text{Id}$, then $f^* = \text{Id}$.

b) If $g: Z \to X$ is continuous then $(fg)^* = g^*f^*$.

c) The construction of f^* is compatible with the construction of Q14, in case f is a homeomorphism and a is the inverse of the natural map $\tilde{f}: f^{-1}F \to F$.

CHAPTER 7. COHERENT SHEAVES

Introduction.

In section 1 we define coherence for sheaves of $\mathcal{O}$-modules and prove Oka's theorem. In section 2 we prove Cartan's theorems A and B for Stein manifolds assuming exactness of the $\bar{\partial}$-sequence. The remainder of the Chapter is devoted to applications of Theorems A and B. We prove Cartan and Serre's finiteness theorem in section 3 and Grauert's finiteness theorem for coherent sheaves on s.L.p. domains in section 4. Using the finiteness theorem of Cartan-Serre, we prove Serre's theorems A and B for coherent sheaves on complex projective space in section 5. We give a number of applications including Grothendieck's theorem on the splitting of holomorphic vector bundles on the Riemann sphere. Finally in section 6 we prove Kodaira's embedding theorem, following Grauert, and conclude by showing that complex tori that admit a Riemann form are algebraic.

§1. Coherent sheaves.

Throughout this section we shall be studying sheaves of $\mathcal{O}$-modules on a complex manifold M (Here, as in the sequel, we usually drop the subscript "M" from the Oka sheaf $\mathcal{O}_M$). In future we call a sheaf of $\mathcal{O}$-modules an *analytic sheaf*. We have already seen some examples where algebraic conditions holding at a point continue to hold in a neighbourhood of the point. Thus the condition that germs of analytic functions be relatively prime is an open condition (Proposition 3.4.2). Another important example came from the theory of analytic hypersurfaces where we showed that if Z was an analytic hypersurface in M with $I_z(Z) = (g_z)$, then $I_y(Z) = (g_y)$ for y in some neighbourhood of z (Theorem 3.5.16). The main aim of this section will be to describe the class of coherent analytic sheaves for which this type of behaviour is characteristic. For example, we shall show that if $F \xrightarrow{a} G \xrightarrow{b} H$ is a sequence of coherent sheaves with ba = 0 and if the sequence is exact (at stalk level) at $x \in M$, then it is exact in a neighbourhood of x. Of course, if we wish to use topological methods to get global results then our local results need to be framed in terms of open sets - not points - and this is why coherence is such an important concept in the cohomology theory of sheaves. Our proofs will be close to those in the original works on coherence by H. Cartan [1,2] and J.P. Serre [2].

Definition 7.1.1. An analytic sheaf F on M is said to be of *finite type* if we can find an open neighbourhood U of every point x in M and a finite number of sections $s_1,\ldots,s_k \in F(U)$ such that $\{s_{1,y},\ldots,s_{k,y}\}$ generates F_y as an $\mathcal{O}_y$-module for every $y \in U$.

Example 1. The sheaf of sections of a holomorphic vector bundle on M is of finite type.

Proposition 7.1.2. Let F be an analytic sheaf of finite type and $f_1,\ldots,f_p$ be continuous sections of F defined over some open neighbourhood of $x \in M$ such that $f_{1,x},\ldots,f_{p,x}$ generate F_x. Then $f_{1,y},\ldots,f_{p,y}$ generate F_y for y in some neighbourhood of x.

Proof. Since F is of finite type we can find an open neighbourhood U of x and sections $s_1,\ldots,s_k \in F(U)$ such that $s_{1,y},\ldots,s_{k,y}$ generate F_y for $y \in U$. By assumption, there exist $a_{ij,x} \in \mathcal{O}_x$ such that

$$s_{i,x} = \sum_{j=1}^{p} a_{ij,x} f_{j,x},\ i = 1,\ldots,k .$$

Choosing representatives for the germs $a_{ij,x}$, we see that on some neighbourhood V of x we have

$$s_i = \sum_{j=1}^{p} a_{ij} f_j,\ i = 1,\ldots,k.$$

Hence for all $z \in V$ we have

$$s_{i,z} = \sum_{j=1}^{p} a_{ij,z} f_{j,z} . \qquad \square$$

For $q \geq 0$, recall that $\mathcal{O}^q$ denotes the p-fole direct sum of $\mathcal{O}$ and that $\mathcal{O}^q$ is equal to the sheaf of germs of $\mathbb{C}^q$-valued holomorphic functions on M. We have a canonical $\mathcal{O}$-module basis of $\mathcal{O}^q$ given by the constant functions

$$E_1(z) = (1,0,\ldots,0),\ldots,E_q(z) = (0,0,\ldots,1).$$

Suppose that F is an analytic sheaf on M and that $f_1,\ldots,f_p$ are continuous sections of F over some open subset U of M. Let $f = (f_1,\ldots,f_p)\colon \mathcal{O}_U^p \to F_U$ denote the sheaf homomorphism defined by

$$f(g_1,\ldots,g_p) = \sum_{j=1}^{p} g_j f_{j,z}, \quad (g_1,\ldots,g_p) \in \mathcal{O}_z^p, \ z \in U.$$

The kernel of f is a subsheaf $R(f_1,\ldots,f_p)$ of $\mathcal{O}_U^p$ called the *sheaf of relations between* $f_1,\ldots,f_p$.

Theorem 7.1.3. Let U be an open subset of M and $f_1,\ldots,f_p \in \mathcal{O}^q(U)$. Then the sheaf of relations $R(f_1,\ldots,f_p)$ is a subsheaf of $\mathcal{O}_U^p$ of finite type.

This theorem, due to Oka, is the fundamental result upon which the theory of coherent analytic sheaves rests.

In view of the local nature of Theorem 7.1.3 it suffices to prove

Theorem 7.1.4. Let Ω be an open subset of $\mathbb{C}^n$ and $f_1,\ldots,f_p \in A(\Omega)^q$. We can find an open neighbourhood U of any point $z \in \Omega$ and finitely many functions $C_1,\ldots,C_r \in A(U)^p$ such that $C_{1,x},\ldots,C_{r,x}$ generate $R(f_1,\ldots,f_p)_x$ as an $\mathcal{O}_x$-module for all $x \in U$.

Proof. First of all notice that

$$R(f_1,\ldots,f_p)_z = \{(a_1,\ldots,a_p) \in \mathcal{O}_z^p\colon \sum_{j=1}^{p} a_j f_{j,z} = 0\}, \quad z \in \Omega.$$

Since $\mathcal{O}_z$ is Noetherian, $R(f_1,\ldots,f_p)_z$ is certainly finitely generated. We have to find generators which also generate the stalks in a neighbourhood of z.

We may clearly assume that $0 \in \Omega$ and that $z = 0$. Our proof goes by induction on q and n. First, suppose that the theorem has been proved for all q in the (n-1)-dimensional case (note that the case $n = 0$ is trivial). We shall prove that the theorem is true for n and $q = 1$.

Without loss of generality we may assume that $f_1,\ldots,f_p$ are normalised in direction z_n and so, by the Weierstrass Preparation theorem, we may suppose that $f_1,\ldots,f_p$ are Weierstrass polynomials in

z_n with coefficients in $A(\Omega')$, where Ω' is an open neighbourhood of $0 \in \mathbb{C}^{n-1}$. In what follows we may suppose $\Omega = \Omega' \times \mathbb{C}$. Let d denote the maximum of the degrees of the polynomials $f_1,\ldots,f_p$. We may write

$$f_i = \sum_{j=0}^{d} f_{ij}z_n^j, \; i = 1,\ldots,p,$$

where $f_{ij} \in A(\Omega')$. We let ${}^dA'[z_n]$ denote the group of polynomials in z_n of degree $\leq d$ which have coefficients in $A(\Omega')$. For $\zeta = (\zeta',\zeta_n) \in \mathbb{C}^n$, we let $\mathcal{O}'_{\zeta'}$ denote the ring of germs of analytic functions on $\mathbb{C}^{n-1}$ at ζ' and ${}^dA'[z_n]_\zeta$ denote the germs at ζ of functions in ${}^dA'[z_n]$. Finally, let ${}^d\mathcal{R}(f_1,\ldots,f_p)$ denote the subset of $\mathcal{R}(f_1,\ldots,f_p)$ defined by ${}^d\mathcal{R}(f_1,\ldots,f_p)_\zeta = \mathcal{R}(f_1,\ldots,f_p)_\zeta \cap {}^dA'[z_n]_\zeta$, $\zeta \in \Omega$. Note that ${}^d\mathcal{R}(f_1,\ldots,f_p)$ has the structure of an $\mathcal{O}'$-module.

We shall prove the inductive step by first showing that for all $\zeta \in \Omega$, ${}^d\mathcal{R}(f_1,\ldots,f_p)_\zeta$ generates $\mathcal{R}(f_1,\ldots,f_p)_\zeta$ as an $\mathcal{O}_\zeta$-module. Then, using the inductive hypothesis, we show that we can find a finite set of generators for ${}^d\mathcal{R}(f_1,\ldots,f_p)$ (as an $\mathcal{O}'$-module) over some neighbourhood of zero.

Step 1: The $\mathcal{O}$-module $\mathcal{R}(f_1,\ldots,f_p)_\zeta$ is generated by the elements of ${}^d\mathcal{R}(f_1,\ldots,f_p)_\zeta$, $\zeta \in \Omega$.

Suppose f_p has degree d. Then, by the Weierstrass Preparation theorem, we have for $\zeta \in \Omega$

$$f_{p,\zeta} = f'f'',$$

where $f', f'' \in \mathcal{O}_\zeta$, f' is the germ of a Weierstrass polynomial in $z_n - \zeta_n$ and $f''(\zeta) \neq 0$. By Lemma 3.4.1, f'' is a polynomial in z_n with leading coefficient 1. Let d', d'' denote the degrees of f', f'' with respect to z_n. Given $(a_1,\ldots,a_p) \in \mathcal{R}(f_1,\ldots,f_p)$, we can by the Weierstrass division theorem write

$$a_i = f_{p,\zeta}b_i + c_i, \; i = 1,\ldots,p-1,$$

where $b_i \in \mathcal{O}_\zeta$ and $c_i \in \mathcal{O}'_{\zeta'}[z_n]$ is of degree $< d'$. Set

$$c_p = a_p + \sum_{i=1}^{p-1} f_{i,\zeta}b_i$$

and observe that we have the identity

$$(a_1,\dots,a_p) = (f_p,0,\dots,0,-f_1)_\zeta b_1 + \dots + (0,\dots,0,f_p,-f_{p-1})_\zeta b_p$$
$$+ (c_1,\dots,c_p) \qquad \dots.(*)$$

Now all the terms in this identity, except possibly $(c_1,\dots,c_p)$, lie in $R(f_1,\dots,f_p)_\zeta$. Hence $(c_1,\dots,c_p) \in R(f_1,\dots,f_p)_\zeta$. Therefore

$$\sum_{i=1}^{p-1} c_i f_{i,\zeta} + (c_p f'')f' = 0.$$

But the sum is a polynomial in z_n of degree $< d+d'$. Consequently, by Lemma 3.4.1, $c_p f''$ is a polynomial in z_n of degree $< d$. But since

$$(c_1,\dots,c_p) = (1/f'')(f''c_1,\dots,f''c_p)$$

and $f''c_1,\dots,f''c_p$ have degree $< d$, it follows that $(c_1,\dots,c_p) \in {}^dR(f_1,\dots,f_p)_\zeta$. This, together with (*), proves step 1.

Step 2: We may find an open neighbourhood U of $0 \in \mathbb{C}^n$ and sections $C_1,\dots,C_r \in {}^dR(f_1,\dots,f_p)(U)$ such that for all $\zeta \in U$, $C_{1,\zeta},\dots,C_{r,\zeta}$ generate ${}^dR(f_1,\dots,f_p)_\zeta$ as an $O'_{\zeta'}$-module.

Suppose $a = (a_1,\dots,a_p) \in {}^dA'[z_n]_\zeta$. Then

$$a_i = \sum_{j=0}^{d} c_{ij}(z_n^j)_\zeta,\ c_{ij} \in O'_{\zeta'}.$$

Now $a \in {}^dR(f_1,\dots,f_p)_\zeta$ if and only if $\sum_{i=0}^{p} a_i f_i = 0$. That is, if and only if

$$\sum_{k=0}^{d}\sum_{j=0}^{d}\sum_{i=1}^{p} c_{ij} f_{ik}(z_n^{k+j}) = 0.$$

Equating coefficients of powers of z_n to zero we see that $a \in {}^dR(f_1,\dots,f_p)_\zeta$ if and only if

$$\sum_{j+k=r}\sum_{i=1}^{p} c_{ij} f_{ik} = 0,\ r = 0,\dots,2d.$$

That is, ${}^dR(f_1,\dots,f_p)$ is isomorphic to the kernel of the homomorphism $F = (F_1,\dots,F_{p(d+1)}): O'^{p(d+1)} \to O'^{2d+1}$ defined by

$$(F([c_{ij}]))_r = \sum_{i,j} c_{ij} f_{i,r-j},\ r = 0,\ldots,2d.$$

In other words, ${}^dR(f_1,\ldots,f_p)$ is isomorphic to $R(F_1,\ldots,F_{p(d+1)})$ and so, by the inductive hypothesis, we may find an open neighbourhood U' of $0 \in \mathbb{C}^{n-1}$ and $C_1,\ldots,C_r \in A(U')^{p(d+1)}$ such that $C_{1,\zeta'},\ldots,C_{r,\zeta'}$ generate $R(F_1,\ldots,F_{p(d+1)})_{\zeta'}$ for all $\zeta' \in U'$. Taking $U = U' \times \mathbb{C} \subset \mathbb{C}^n$, we see that $C_1,\ldots,C_r$ give the required generators of ${}^dR(f_1,\ldots,f_p)$ as an $\mathcal{O}'$-module.

To complete our induction we now show that if the result is true for n and $q = 1$ it is true for n and $q > 1$.

Setting $f_j = (f_{j1},\ldots,f_{jq})$, we define $\tilde{f}_j = (f_{j1},\ldots,f_{jq-1})$, $j = 1,\ldots,p$. For $\zeta \in \Omega$, we have

$$R(f_1,\ldots,f_p)_\zeta \subset R(\tilde{f}_1,\ldots,\tilde{f}_p)_\zeta .$$

Now by the inductive hypothesis for n and $q-1$, there exists a neighbourhood $V \subset \Omega$ of 0 and $g_1,\ldots,g_r \in A(V)^p$ such that $g_{1,\zeta},\ldots,g_{r,\zeta}$ generate the $\mathcal{O}_\zeta$-module $R(\tilde{f}_1,\ldots,\tilde{f}_p)_\zeta$ for all $\zeta \in V$. For $\zeta \in V$, we have $R(f_1,\ldots,f_p)_\zeta \subset \left\{\sum_{j=1}^{r} c_j g_{j,\zeta}: c_j \in \mathcal{O}_\zeta\right\}$. Setting $g_j = (g_{j1},\ldots,g_{jp})$, $1 \le j \le r$, and taking components as we see that $\sum_{j=1}^{r} c_j g_{j,\zeta} \in R(f_1,\ldots,f_p)_\zeta$ if and only if

$$\sum_{j=1}^{r}\sum_{k=1}^{p} c_j (g_{jk} f_{ki})_\zeta = 0,\ i = 1,\ldots,q .$$

But the first $q-1$ of these equations automatically hold and so only the qth. equation remains to be satisfied. By the inductive hypothesis for n and $q = 1$, there exists a neighbourhood $U \subset V$ of 0 and $h_1,\ldots,h_s \in A(U)^r$ such that the solutions $(c_1,\ldots,c_r) \in \mathcal{O}_\zeta^r$ of $\sum_{j=1}^{r}\sum_{k=1}^{p} c_j (g_{jk} f_{kq})_\zeta = 0$ are generated by $h_{1,\zeta},\ldots,h_{s,\zeta}$, $\zeta \in U$. Defining $C_1,\ldots,C_s \in A(U)^p$ by $C_i = \sum_{j=1}^{r} h_{ij} g_j$, $1 \le i \le s$, we see that $C_1,\ldots,C_s$ are the required set of generators for $R(f_1,\ldots,f_p)_U$. □

Definition 7.1.5. An analytic sheaf F on the complex manifold M is said to be *coherent* if

a) F is of finite type.

b) Given any open subset U of M and sections $f_1,\dots,f_p \in F(U)$, then the sheaf of relations $R(f_1,\dots,f_p)$ is of finite type.

Theorem 7.1.6. Every analytic subsheaf of O^q which is of finite type is coherent.

Proof. Theorem 7.1.3. □

Corollary 7.1.7. Let F be a coherent sheaf on M, U be an open subset of M and $f_1,\dots,f_p \in F(U)$. Then $R(f_1,\dots,f_p)$ is coherent.

Proof. $R(f_1,\dots,f_p)$ is a subsheaf of O_U^p of finite type. □

Corollary 7.1.8. Let F be a coherent sheaf on M and $x \in M$. Then we may find a free resolution of F, of length $m = \dim(M)$, over some open neighbourhood U of x:

$$0 \to O_U^{p_m} \xrightarrow{s_m} O_U^{p_{m-1}} \to \dots \xrightarrow{s_1} O_U^{p_0} \xrightarrow{s_0} F_U \to 0 .$$

Proof. Since F is of finite type, we may find an open neighbourhood U_0 of x and $s_1^0,\dots,s_{p_0}^0 \in F(U_0)$ such that for all $\zeta \in U_0$, $s_{1,\zeta}^0,\dots,s_{p_0,\zeta}^0$ generate F_ζ as an O_ζ-module. That is, setting $s_0 = (s_1^0,\dots,s_{p_0}^0)$ we have the exact sequence

$$O_{U_0}^{p_0} \xrightarrow{s_0} F_{U_0} \to 0 .$$

Now $\mathrm{Ker}(s_0) = R(s_1^0,\dots,s_{p_0}^0)$ and so, since $R(s_1^0,\dots,s_{p_0}^0)$ is of finite type, we may find an open neighbourhood $U_1 \subset U_0$ of x and $s_1^1,\dots,s_{p_1}^1 \in A(U_1)^{p_0}$ such that $s_{1,\zeta}^1,\dots,s_{p_1,\zeta}^1$ generate $R(s_1^0,\dots,s_{p_0}^0)_\zeta$, $\zeta \in U_1$. Setting $s_1 = (s_1^1,\dots,s_{p_1}^1)$, we obtain the exact sequence

$$O_{U_1}^{p_1} \xrightarrow{s_1} O_{U_1}^{p_0} \xrightarrow{s_0} F_{U_1} \to 0.$$

Proceeding inductively, we obtain after m steps the exact sequence

$$O_{U_{m-1}}^{p_{m-1}} \xrightarrow{s_{m-1}} \dots \xrightarrow{s_1} O_{U_{m-1}}^{p_0} \xrightarrow{s_0} F_{U_{m-1}} \to 0 .$$

By the Hilbert Syzygy theorem (Theorem 3.6.1), $\mathrm{Ker}(s_{m-1})_y$ is a free $\mathcal{O}_y$-module, $y \in U_{m-1}$. Suppose $\mathrm{Ker}(s_{m-1})_x \cong \mathcal{O}_x^{p_m}$. A set of generators for $\mathcal{O}_x^{p_m}$ is given by the constant functions E_j, $1 \le j \le p_m$ and, since $\mathrm{Ker}(s_{m-1})$ is of finite type, it follows from Proposition 7.1.2 that the E_j generate $\mathrm{Ker}(s_{m-1})_y$ for y in some open neighbourhood U_m of x. Taking $U = U_m$, we therefore obtain the exact sequence

$$\mathcal{O}_U^{p_m} \xrightarrow{s_m} \mathcal{O}_U^{p_{m-1}} \to \dots \xrightarrow{s_0} F_U \to 0.$$

By the Hilbert Syzygy theorem, $\mathrm{Ker}(s_m) = 0$. □

Remarks. One consequence of Theorem 7.1.8 is that every coherent sheaf is locally isomorphic to the cokernel of a sheaf homomorphism $a: \mathcal{O}^p \to \mathcal{O}^q$. Later on in this chapter, we shall examine to what extent coherent sheaves admit global resolutions by free, and more expecially locally free, sheaves of $\mathcal{O}$-modules.

Example 2. The sheaf of holomorphic sections of a holomorphic vector bundle E is coherent. Indeed, $\underline{E}$ is locally isomorphic to $\mathcal{O}^q$, $q = \dim(E)$.

Theorem 7.1.9. We have the following basic properties of coherent sheaves:

1. Every analytic subsheaf of a coherent sheaf which is of finite type is coherent.

2. Suppose $0 \to F \xrightarrow{a} G \xrightarrow{b} H \to 0$ is a short exact sequence of analytic sheaves. If any two of the sheaves F, G, H are coherent, so is the third.

3. The direct sum of a finite family of coherent sheaves is coherent.

4. Let $a: F \to G$ be a homomorphism of coherent sheaves. Then Ker(a), Im(a) and Coker(a) are coherent.

5. Let F, G be coherent subsheaves of a coherent sheaf H. Then the sheaves $F + G$, $F \cup G$ are coherent.

6. Let F, G be coherent sheaves. Then $F \otimes_{\mathcal{O}} G$ is coherent.

Proof. 1. Clearly every analytic subsheaf of a sheaf satisfying (b) of Definition 7.1.5 also satisfies condition (b).

2.a) Suppose G and H are coherent. Since G is of finite type, we may find an open neighbourhood U of any point $x \in M$ and surjective homomorphism $c: \mathcal{O}_U^p \to G_U$. Since H satisfies condition (b), Ker(bc) is of finite type and so, by 1., c(Ker(bc)) is a coherent subsheaf of G_U. But a maps F_U isomorphically onto c(Ker(bc)) and so F is coherent.

b) Suppose F and G are coherent. Since G is of finite type and b is surjective H is of finite type. We must show that H satisfies condition (b). Let $x \in M$, U be an open neighbourhood of x and $s_1,\dots,s_p \in H(U)$. Shrinking U if necessary, we may choose $t_1,\dots,t_p \in G(U)$ such that $s_j = b(t_j)$, $j = 1,\dots,p$. Shrinking U further if necessary, we may find $u_1,\dots,u_q \in F(U)$ such that $u_{1,y},\dots,u_{q,y}$ generate F_y as an $\mathcal{O}_y$-module for all $y \in U$. Now given $y \in U$ and $(f_1,\dots,f_p) \in \mathcal{O}_y^p$, $(f_1,\dots,f_p) \in R(s_1,\dots,s_p)_y$ if and only if $\sum_{i=1}^{p} f_i t_i \in \mathrm{Im}(a)$. That is, if and only if there exist $g_1,\dots,g_q \in \mathcal{O}_y$ such that $\sum_{i=1}^{p} f_i t_i = \sum_{j=1}^{q} g_j a(u_j)$. But $R(t_1,\dots,t_p,a(u_1),\dots,a(u_q))$ is of finite type since G is coherent and since $R(s_1,\dots,s_p)$ is the image of $R(t_1,\dots,a(u_q))$ under the canonical projection of $\mathcal{O}^{p+q}$ on $\mathcal{O}^p$ it follows that $R(s_1,\dots,s_p)$ is of finite type.

c) We leave the case F and H coherent implies G coherent as an exercise (details may be found in Serre [2] or Gunning and Rossi [1]).

3. The finite direct sum of coherent sheaves is coherent. This follows by an easy induction from 2.c) or directly and we omit details.

4. Suppose $a: F \to G$ is a homomorphism of coherent sheaves. Now Im(a) is of finite type since F is of finite type and so, by 1, Im(a) is coherent. Applying 2.a) and b) to the exact sequences

$$0 \to \mathrm{Ker}(a) \to F \to \mathrm{Im}(a) \to 0$$

$$0 \to \mathrm{Im}(a) \to G \to \mathrm{Coker}(a) \to 0$$

we see that Ker(a) and Coker(a) are coherent.

5. The sheaf $F + G$ is of finite type and so coherent by 1. The sheaf $F \cap G$ is the kernel of the quotient map $F \to H/G$ and is therefore coherent by 2.a).

6. Let $x \in M$. Since F is coherent, there exists an open neighbourhood U of x and exact sheaf sequence

$$\mathcal{O}_U^p \to \mathcal{O}_U^q \to F_U \to 0.$$

Tensoring with G_U we obtain the exact sequence

$$G_U \otimes_{\mathcal{O}} \mathcal{O}_U^p \to G_U \otimes_{\mathcal{O}} \mathcal{O}_U^q \to (F \otimes_{\mathcal{O}} G)_U \to 0.$$

Now $G_U \otimes_{\mathcal{O}} \mathcal{O}_U^p \approx G_U^p$; $G_U \otimes_{\mathcal{O}} \mathcal{O}_U^q \approx G_U^q$ and so we arrive at the exact sequence

$$G_U^p \to G_U^q \to (F \otimes_{\mathcal{O}} G)_U \to 0.$$

But by 3, G_U^p, G_U^q are coherent and so, by 4, $(F \otimes_{\mathcal{O}} G)_U$ is coherent. Hence $F \otimes_{\mathcal{O}} G$ is coherent. □

Remark. Theorem 7.1.9 suggests that set of coherent sheaves on M is the smallest class of analytic sheaves on M which contains the locally free sheaves (holomorphic vector bundles) and is closed under the operations of quotient, kernel and image. Unfortunately this is not generally true, even when M is compact (the ideal sheaf of a point need not have a resolution by locally free sheaves in case every holomorphic vector bundle on M is flat). However, if M is projective algebraic, then every coherent sheaf on M has a global resolution by locally free sheaves and so the coherent sheaves on M are the smallest class containing the locally free sheaves and which are closed under the operations of quotient, kernel and image. We shall return to this question later in the chapter.

Examples.

3. Let X be a complex submanifold of the complex manifold M. Then the sheaves I_X, $\mathcal{O}_X$ are coherent sheaves on M. First we prove I_X is coherent. The question being local we may suppose that X is the subspace $\mathbb{C}^k$ of $\mathbb{C}^m$ defined by $z_{k+1} = \dots = z_m = 0$. If $z \notin \mathbb{C}^k$, $I_{X,z} = \mathcal{O}_z$.

For $z \in \mathbb{C}^k$, $I_{X,z} = (z_{k+1},\ldots,z_m)$ and clearly $z_{k+1},\ldots,z_m$ give a finite set of generators for I_X in a neighbourhood of z. Hence I_X is of finite type and so coherent by Theorem 7.1.6. Since $O_X = O_M/I_X$, we see from Theorem 7.1.9, 2, that O_X is coherent.

4. Let X be an analytic hypersurface in the complex manifold M. Then I_X, O_X are coherent. Theorem 3.5.16 implies that I_X is coherent. O_X is coherent as in Example 3.

5. Let X be an analytic subset of M. Then I_X, O_X are coherent. The proof of this result is outside the scope of these notes depending, as it does, on the local parametrization theorem for analytic sets. The proof may be found in H. Cartan [1], Gunning [1 ; page 43], Gunning and Rossi [1], R. Narasimhan [1], Whitney [1].

6. Let $f\colon M \to N$ be a holomorphic map of complex manifolds and F be a coherent sheaf on N. Then f^*F is a coherent sheaf on M. This result follows by representing F locally as the cokernel of a map $O^p \to O^q$ and then using the right exactness of f^* - see Exercise 12, §1, Chapter 6.

7. Suppose F is a coherent sheaf on M and $f\colon M \to N$ is holomorphic. In general f_*F will not be a coherent sheaf on N. However, if f is proper, a deep and difficult theorem of Grauert asserts that f_*F is coherent. Proofs of this important result may be found in Foster and Knorr [1], Kiehl and Verdier [1]. See also R. Narasimhan [2]. In case f is a *finite* map, the reader may consult Grauert and Remmert [1], Gunning [1], R. Narasimhan [1].

8. Let F be a coherent sheaf on M. Then supp(F) is an analytic subset of M (for the definition of supp(F), see Exercise 6, §1, Chapter 6). To see this note that we can find an open neighbourhood U of any $x \in M$ and exact sequence

$$O_U^p \xrightarrow{s_0} O_U^q \xrightarrow{s_1} F_U \to 0 .$$

Now $\mathrm{supp}(F) \cap U = \{x \in U\colon s_{1,x} \neq 0\} = \{x \in U\colon s_{0,x}$ is not of maximal rank$\}$. Now s_0 may be represented as a $q \times p$ matrix with holomorphic entries defined on U. The condition that $s_{0,x}$ is not of maximal rank is given by a finite set of algebraic equations in the components of $s_{0,x}$. Hence the set of points where $s_{0,x}$ fails to be of maximal rank is an analytic subset of U. A similar argument will show that the set of points in M where F is not locally free is an analytic subset of M.

9. Let X be a complex submanifold of M and let $\mathcal{O}_X$ denote $\mathcal{O}_X|X$ as well as the sheaf $\mathcal{O}_X$ on M. Suppose F is a sheaf of $\mathcal{O}_X$-modules on X and $\tilde{F}$ denote the trivial extension of F to M. Clearly $\tilde{F}$ has the structure of an $\mathcal{O}_X$-module and, since $\mathcal{O}_X$ is an $\mathcal{O}_M$-module, $\tilde{F}$ has the structure of an $\mathcal{O}_M$-moudle. We claim that $\tilde{F}$ is coherent as a sheaf of $\mathcal{O}_M$-modules if and only if F is coherent as a sheaf of $\mathcal{O}_X$-modules. Well suppose F is coherent as a sheaf of $\mathcal{O}_X$-modules. Obviously $\tilde{F}|M \setminus X$ is coherent. Let $x \in X$. We may find an open neighbourhood U of x in X and exact sequence

$$\mathcal{O}_U^q \to \mathcal{O}_U^p \to F_U \to 0.$$

Taking any open neighbourhood V of x in M such that $V \cap X = U$ and taking trivial extensions we obtain the exact sequence

$$(\mathcal{O}_X|V)^q \to (\mathcal{O}_X|V)^p \to \tilde{F}_V \to 0 .$$

But $\mathcal{O}_X^p$, $\mathcal{O}_X^q$ are coherent sheaves on M and so by Theorem 7.1.9, 4, $\tilde{F}_V$ is coherent. Hence $\tilde{F}$ is coherent. The converse follows easily from Example 6.

10. Let $F \xrightarrow{a} G \xrightarrow{b} H$ be a sequence of coherent sheaves on M and suppose $ba = 0$ and the sequence is exact at the point $x \in M$ (that is at the stalk level). Then the sequence is exact on some open neighbourhood of x. To see this we note that ker(b)/Im(a) is coherent with zero stalk at x. Now apply the result of Example 8.

Exercises.

1. Let F, G be analytic sheaves on X with F coherent. Show that for all $x \in X$, $\mathcal{H}om_{\mathcal{O}}(F,G)_x \approx \mathrm{Hom}_{\mathcal{O}_x}(F_x,G_x)$. Deduce that if F and G are coherent then so is $\mathcal{H}om_{\mathcal{O}}(F,G)$.

2. Remmert's proper mapping theorem states that the image of an analytic subset by a proper analytic map is an analytic set. Deduce this result from Grauert's direct image theorem and Example 8.

3. Let $(X,\mathcal{O}_X)$ be an analytic space (§1, Chapter 6). Using the result described in Example 5, show that $\mathcal{O}_X^p$ is a coherent sheaf of $\mathcal{O}_X$-modules, $p \geq 1$. Describe the appropriate extension of Example 9 to this more general framework.

4. Show that $f_*\mathcal{F}$ need not be of finite type if f is not proper, $\mathcal{F}$ coherent. (Hint: Take $X = \mathbb{C}$, Y a point and $\mathcal{F} = \mathcal{O}_X$).

5. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on the complex manifold X. Suppose that $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for every $x \in X$. Prove that $\mathcal{F}$ is locally free.

§2. Coherent sheaves on a Stein manifold.

Suppose that E is a holomorphic vector bundle on the m-dimensional complex manifold M. We have the associated $\bar\partial$-complex

$$0 \to \Omega(E) \to C^{0,0}(M,E) \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} C^{0,m}(M,E) \to 0.$$

For this section and the remainder of the chapter we shall assume that the $\bar\partial$-complex of any holomorphic vector bundle on a Stein manifold is *exact*. In Chapter 11 we shall give a proof of this fundamental result that depends on the theory of elliptic operators. For the present we remark that we have proved the exactness of the $\bar\partial$-sequence in case M is a polydisc or $\mathbb{C}^n$ and E is trivial (Theorem 5.8.2) and indicated an elementary proof in case M is the Euclidean disc and E is trivial (Exercise, 2, §8, Chapter 5).

The Dolbeault isomorphisms (Example 5, §3, Chapter 6) imply that our assumption is equivalent to the vanishing of $H^p(M,\underline{E})$, $p \geq 1$, for every holomorphic vector bundle E on a Stein manifold M. Since the sheaf of sections of a holomorphic vector bundle is coherent we see that our assumption amounts to a special case of the second of the remarkable and famous Theorems A and B of H. Cartan:

Theorem A. Let F be a coherent sheaf on the Stein manifold M. Given $x \in M$, we may find $s_1,\dots,s_p \in H^0(M,F) = F(M)$ such that $s_{1,x},\dots,s_{p,x}$ generate F_x as an O_x-module.

Theorem B. Let F be a coherent sheaf on the Stein manifold M. Then $H^q(M,F) = 0$, $q \geq 1$.

The main aim of this section is to prove Theorems A and B, granted our assumption that Theorem B is true for locally free sheaves of O-modules.

Theorems A and B have many profound applications.

Examples.

1. Let X be an analytic subset of the Stein manifold M. Then X may be represented as the common zero locus of a set of analytic functions on M. For this it is enough to show that for each $x \in M \setminus X$, there exists $f \in A(M)$ such that $f \in I_X(M)$ and $f(x) \neq 0$. Set $I = I_X$ and let I_x denote the ideal sheaf of the analytic set $X \cup \{x\}$. Then I_x is a subsheaf of O and $I/I_x = \mathbb{C}(x)$, where $\mathbb{C}(x)$ denotes the "skyscrapper" sheaf whose stalk is zero except at x where it equals $\mathbb{C}$. Now, by Cartan's coherence theorem, I_x is coherent and so, by Theorem B, $H^1(M,I_x) = 0$. Therefore, taking the cohomology sequence of the short exact sequence $0 \to I_x \to I \to \mathbb{C}(x) \to 0$ we obtain the exact sequence

$$I(M) \to \mathbb{C} \to 0.$$

Hence, for any $a \in \mathbb{C}$, there exists $f \in I(M) \subset A(M)$ such that $f(x) = a$. Choosing $a \neq 0$, our proof is complete. In fact a much sharper result is true. It can be shown that if M is of dimension m, there exist $f_1,\dots,f_m \in A(M)$ such that $X = Z(f_1,\dots,f_m)$. The reader may find a proof in Forster and Ramspott [1] (see also Grauert [2]).

2. Let M be a complex manifold and $U = \{U_i : i \in I\}$ be an open cover of M by Stein manifolds. Then, exactly as in the proof of Example 15, §4, Chapter 2, $U_{i_0\dots i_k} = U_{i_0} \cap \dots \cap U_{i_k}$ is Stein for all $i_0,\dots,i_k \in I$. It follows from Theorem B that if F is any coherent sheaf on M, then any Stein open cover of M is a Leray cover for F.

Before we start on the main work of this section, we show how Theorem B for free sheaves enables us to give a complete solution to the Cousin problems on a Stein manifold.

Theorem 7.2.1. A Stein manifold is a Cousin I and Cousin A domain.

Proof. Our standing assumption implies that $H^1(M,\mathcal{O})$ $(= H^1(M,\underline{\mathbb{C}})) = 0$, M Stein. Hence, as in Example 6, §3, Chapter 6, M is a Cousin I and Cousin A domain. □

Theorem 7.2.2. Let d be a divisor on the Stein manifold M. Then d is a divisor of a meromorphic function on M if and only if $c_1([d]) = 0$. In particular, M is a Cousin I and Cousin B domain if and only if $H^2(M, \mathbb{Z}) = 0$.

Proof. Our standing assumption implies that $H^1(M,\mathcal{O}) = H^2(M,\mathcal{O}) = 0$. Hence the Chern class map $c_1: H^1(M,\mathcal{O}^*) \to H^2(M,\mathbb{Z})$ is an isomorphism (see Example 11, §3, Chapter 6). The result now follows from Example 16, §3, Chapter 6.

As an immediate consequence of Theorem 7.2.2 we have

Theorem 7.2.3. Let X be an analytic hypersurface in the Stein manifold M. Provided that $H^2(M,\mathbb{Z}) = 0$, there exists $f \in A(M)$ such that

1. $X = Z(f)$.
2. $\mathcal{I}_{X,x} = (f_x)$, for all $x \in M$.

Remark. If X is an analytic hypersurface in the m-dimensional Stein manifold M and $H^2(M,\mathbb{Z}) \neq 0$, then it can be shown that X is representable as the common zero locus of not more than $1 + [\frac{m}{2}]$ analytic functions on M. See Forster and Ramspott [1; Satz 3].

We shall give further applications of Theorems A and B in the remainder of the chapter and also in Chapter 12.

Proposition 7.2.4. Let $\mathcal{F}$ be a coherent sheaf on the Stein manifold M. Suppose that $\mathcal{F}$ admits the (projective) resolution

$$0 \to \underline{E}_m \xrightarrow{a_m} \ldots . \xrightarrow{a_1} \underline{E}_0 \xrightarrow{a_0} F \to 0,$$

where $E_m,\ldots,E_0$ are holomorphic vector bundles on M. Then

1. $H^q(M,F) = 0$, $q \geq 1$.

2. The map $a_0^0\colon \Omega(E_0) \to F(M)$ is onto.

Proof. Set $K_j = \operatorname{Im}(a_j)$, $0 \leq j \leq m$. The exactness of the resolution of F is equivalent to the exactness of the sequences

$$0 \to K_{j+1} \to \underline{E}_j \to K_j \to 0,\ j = 0,\ldots,m-1.$$

Our standing assumption implies that $H^q(M,\underline{E}_j) = 0$, $q \geq 1$, $0 \leq j \leq m$. Consider the cohomology sequence of the short exact sequence $0 \to K_m \to \underline{E}_{m-1} \to K_{m-1} \to 0$. Since $K_m \cong \underline{E}_m$, we see easily that $H^q(M,K_{m-1}) = 0$, $q \geq 1$. Proceeding inductively, we deduce that $H^q(M,K_j) = 0$, $q \geq 1$, $j = 0,\ldots,m-1$. But $K_0 = F$ and so we have proved 1. Now take the cohomology sequence of $0 \to K_1 \to \underline{E}_0 \xrightarrow{a_0} F \to 0$ to obtain the exact sequence

$$0 \to K_1(M) \to \Omega(E_0) \xrightarrow{a_0^0} F(M) \to 0.$$

This proves 2. □

Examples.

3. Let X be an analytic hypersurface in the Stein manifold M and suppose that $H^2(M,\mathbb{Z}) = 0$. We claim that $H^q(M,\mathcal{O}_X) = 0$, $q \geq 1$. Indeed by Theorem 7.2.3, there exists $f \in A(M)$ such that $I_{X,x} = (f_x)$ for all $x \in M$. Therefore $\mathcal{O}_X$ has the resolution

$$0 \to \mathcal{O}_M \xrightarrow{\times f} \mathcal{O}_M \to \mathcal{O}_M/I_X = \mathcal{O}_X \to 0.$$

Now apply Proposition 7.2.4.

4. Let I_0 denote the ideal sheaf of the point $(0,0) \in \mathbb{C}^2$ and set $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2}$, $\mathcal{O}_0 = \mathcal{O}/I_0$. Then $\mathcal{O}_0$ has the resolution

$$0 \to \mathcal{O} \xrightarrow{a} \mathcal{O}^2 \xrightarrow{b} \mathcal{O} \to \mathcal{O}_0 \to 0,$$

where $a(f) = (z_2f, z_1f)$, $b(f_1, f_2) = z_1f_1 - z_2f_2$.
Hence, by Proposition 7.2.4, $H^q(\mathbb{C}^2, \mathcal{O}_0) = 0$, $q \geq 1$.

Remarks.

1. Both the examples above are special instances of a general technique for constructing resolutions of ideal or structure sheaves based on the *Koszul complex*. For details we refer the reader to Griffiths and Harris [1] and Hartshorne [1]. See also the exercises at the end of §5.

2. Unfortunately it is not true that every coherent sheaf on a Stein manifold admits a resolution by free or even locally free sheaves (see exercise 9). However, we shall prove that a coherent sheaf on a Stein manifold M admits a free resolution over any relatively compact subset of M and this will be a main step in the proof of Theorems A and B.

The next few paragraphs are devoted to a study of the space of sections of a holomorphic vector bundle defined over a domain of holomorphy.

Proposition 7.2.5. Let E be a holomorphic vector bundle over the domain of holomorphy Ω in $\mathbb{C}^n$. Then

1. Given $e \in E_x$, $x \in \Omega$, there exists $s \in \Omega(E)$ such that $s(x) = e$.
2. If ω is any relatively compact open subset of Ω, there exists an exact sequence

$$\mathcal{O}^r_\omega \to \underline{E}_\omega \to 0.$$

Proof. Our proof of 1 goes by induction on n. Suppose $n = 1$. Let $\mathcal{I}$ denote the ideal sheaf of $\{x\}$. As in example 3 we have the free resolution $0 \to \mathcal{O} \xrightarrow{a} \mathcal{O} \to 0$ of $\mathcal{I}$, where a is defined as multiplication by $z - x$. Tensoring with $\underline{E}$, we obtain the exact sequence

$$0 \to \underline{E} \to \underline{E}^x \to 0,$$

where $\underline{E}^x = I \otimes_{\mathcal{O}} \underline{E}$ and is the subsheaf of sections of $\underline{E}$ which vanish at x. Taking the cohomology sequence and applying our standing assumption we see that $H^p(\Omega, \underline{E}^x) = 0$, $p \geq 1$. Since $\underline{E}^x$ is a subsheaf of $\underline{E}$, we have the short exact sequence

$$0 \to \underline{E}^x \to \underline{E} \xrightarrow{P} E(x) \to 0$$

where E(x) is the skyscrapper sheaf with stalk E_x at x and zero stalk everywhere else and P evaluates sections at x. Taking the cohomology sequence and using the vanishing of $H^1(\Omega, \underline{E}^x)$, we arrive at the exact sequence

$$0 \to \Omega(\underline{E}^x) \to \Omega(E) \xrightarrow{P} E_x \to 0$$

and so P is onto, proving 1. Now suppose the result proven for n - 1. Without loss of generality suppose $x = 0 \in \Omega$. Let H denote the intersection of the hyperplane $z_1 = 0$ with Ω. Since H is obviously holomorphically convex, H is a domain of holomorphy (in $\mathbb{C}^{n-1}$). We let $\mathcal{O}'$ denote the Oka sheaf of H and remark that $\mathcal{O}' \cong {}_{n-1}\mathcal{O}|H$. We have the short exact sequence $0 \to \mathcal{O} \xrightarrow{a} \mathcal{O} \xrightarrow{r} \mathcal{O}' \to 0$ of sheaves over Ω where a corresponds to multiplication by z_1 and r is restriction of germs to H. Tensoring this sequence with $\underline{E}$ we obtain the ahort exact sequence

$$0 \to \underline{E} \to \underline{E} \xrightarrow{r} \underline{E}_H \to 0$$

where $\underline{E}_H$ is the sheaf of sections of the holomorphic bundle E restricted to H. Taking the cohomology sequence we arrive at the short exact sequence

$$0 \to \Omega(E) \to \Omega(E) \xrightarrow{r} \Omega(E|H) \to 0.$$

By our inductive assumption, there exists $\tilde{s} \in \Omega(E|H)$ such that $s(x) = e$. Since r is onto, it follows that there exists $s \in \Omega(E)$ such that $s(x) = e$, completing the inductive step and proving 1.

For the proof of 2 we note that for each $x \in \omega$, there exist, by 1, $s_1, \ldots, s_q \in \Omega(E)$ such that $\{s_1(x), \ldots, s_q(x)\}$ is a basis for E_x

$q = \dim(E)$. By the compactness of $\bar{\omega}$, we may therefore find sections $s_1^1,\dots,s_q^1,s_1^2,\dots,s_q^p \in \Omega(E)$ such that for any $x \in \bar{\omega}$, $\{s_1^1(x),\dots,s_q^p(x)\}$ spans E_x. Taking $r = pq$, we see that the map $c = (s_1^1,\dots,s_q^p): \mathcal{O}^r \to \underline{E}$ is onto. □

The next theorem provides a key step towards the proof of Theorem B.

Theorem 7.2.6. Let F be a coherent sheaf on the Stein manifold M. Then there exists a free resolution of F over any relatively compact open subset ω of M:

$$0 \to \mathcal{O}_\omega^{p_m} \to \dots \to \mathcal{O}_\omega^{p_0} \to F \to 0.$$

Our proof of Theorem 7.2.6 will depend on several lemmas. Essentially, we shall first prove the theorem in case M is a domain of holomorphy (hard) and then, using a lemma of Rossi, deduce the general case (easy).

Definition 7.2.7. An open subset P of the Stein manifold M is called an *analytic polyhedron* if there exist $f_1,\dots,f_k \in A(M)$ such that P is a union of connected components of the set $\{z \in M: |f_j(z)| < 1, j = 1,\dots,k\}$.

Remark. Just as in §4, Chapter 2, it is easily verified that an analytic polyhedron is holomorphically convex and therefore a Stein manifold.

Lemma 7.2.8. Let U be an open neighbourhood of the compact subset K of the Stein manifold M. Suppose that K is A(M)-convex (that is, $\hat{K} = K$). Then we may find an analytic polyhedron $P \subset U$ which is a neighbourhood of K.

Proof. Without loss of generality we may suppose that U is relatively compact. For each $x \in \partial U$, there exists $F \in A(M)$ such that $|F(x)| > 1$, $\|F\|_K < 1$. By the compactness of ∂U, we may therefore find $F_1,\dots,F_q \in A(M)$ such that $\|F_1\|_K,\dots,\|F_q\|_K < 1$ and $\max|F_j(x)| > 1$ for every $x \in \partial U$. Since $Q = \{z \in M: |F_j(z)| < 1, j = 1,\dots,q\}$ is disjoint from ∂U, we may take $P = Q \cap U$. □

Suppose K is a compact A(M)-convex subset of the Stein manifold M. We say that K possesses property (R) if given any open neighbourhood U of K and coherent sheaf F on U, we can find an open neighbourhood V of K contained in U and exact sequence

$$\mathcal{O}_V^p \to F_V \to 0$$

(V and p will depend on F).

Lemma 7.2.9. Suppose that the compact A(M)-convex subset K possesses property (R). Then given an open neighbourhood U of K and coherent sheaf F on U, we can find a Stein open neighbourhood $\omega \subset U$ of K and free resolution of F over ω:

$$0 \to \mathcal{O}_\omega^{p_m} \to \dots \to \mathcal{O}_\omega^{p_0} \xrightarrow{a_0} F_\omega \to 0.$$

Proof. Since K possesses property (R), we may find an open neighbourhood $U_0 \subset U$ of K and exact sequence $\mathcal{O}_{U_0}^{p_0} \xrightarrow{a_0} F_{U_0} \to 0$. Now $\mathrm{Ker}(a_0)$ is a coherent sheaf on U_0 and so, since K possesses property (R), we have an exact sequence $\mathcal{O}_{U_1}^{p_1} \xrightarrow{a_1} \mathrm{Ker}(a_0)_{U_1} \to 0$ over some open neighbourhood $U_1 \subset U_0$ of K. That is, we have the exact sequence $\mathcal{O}_{U_1}^{p_1} \xrightarrow{a_1} \mathcal{O}_{U_1}^{p_0} \xrightarrow{a_0} F_{U_1} \to 0$. We now proceed inductively, just as in the proof of Corollary 7.1.8, to obtain a free resolution

$$0 \to \mathcal{O}_{U_m}^{p_m} \to \dots \to \mathcal{O}_{U_m}^{p_0} \to F_{U_m} \to 0 \qquad \dots.(*)$$

of F over some open neighbourhood $U_m \subset U$ of K. Finally, by Lemma 7.2.8 we may find a Stein neighbourhood $\omega \subset U_m$ of K and restricting (*) to ω we obtain the required result. □

Corollary 7.2.10. Suppose that the compact A(M)-convex compact subset K possesses property (R) and that F is a coherent sheaf defined on some open neighbourhood of K. Then

1. For every open neighbourhood V of K, we may find a Stein open neighbourhood $\omega \subset V$ of K such that $H^q(\omega, F_\omega) = 0$, $q \geq 1$.

2. If $f, f_1, \ldots, f_k$ are sections of F over some open neighbourhood V of K and if $f_1, \ldots, f_k$ generate F_V, we may find a (Stein) open neighbourhood $\omega \subset V$ of K and $a_1, \ldots, a_k \in A(\omega)$ such that

$$f = \sum_{j=1}^{k} a_j f_j \text{ on } \omega.$$

Proof. Both 1 and 2 are immediate from Lemma 7.2.1 and Proposition 7.2.4. □

We now come to our main lemma.

Lemma 7.2.11 (H. Cartan). Let K be a compact $A(\Omega)$-convex subset of the domain of holomorphy Ω. Suppose that $f \in A(\Omega)$ and that $K_a = \{z \in K: \mathrm{Re}(f(z)) = a\}$ possesses property (R) for all $a \in \mathbb{R}$. Then K possesses property (R).

Proof. Our proof is a combination of that due to H. Cartan together with a twist due to Hörmander [1] which makes use of Theorem B for locally free sheaves.

Let F be a coherent sheaf defined on some neighbourhood of K. For $-\infty \le a \le b \le +\infty$ we set $K_{a,b} = \{z \in K: a \le \mathrm{Re}(f(z)) \le b\}$. Certainly $K_{a,b}$ is $A(\Omega)$-convex since the condition $a \le \mathrm{Re}(f(z)) \le b$ may be equivalently written as $|\exp(f(z))| \le \exp(b)$; $|\exp(-f(z))| \le \exp(-a)$.

For sufficiently large negative numbers a, $K_{-\infty,a} = \emptyset$ and so $K_{-\infty,a}$ possesses property (R). Let S denote the supremum of numbers a such that $K_{-\infty,a}$ possesses property (R). It is enough to prove $S = +\infty$, since $K_{-\infty,+\infty} = K$.

Suppose $S < +\infty$. Since $K_{S,S} = K_S$ possesses property (R), Proposition 7.2.4 and Corollary 7.2.10 imply that there exists a Stein open neighbourhood U_1 of K_S and $f_1 = (f_1^1, \ldots, f_1^p) \in F(U_1)^p$, such that $f_1^1, \ldots, f_1^p$ generate F as an O-module over U_1. Choose $a < S < b$ so that $K_{a,b} \subset U_1$. By definition of S and Proposition 7.2.4, Corollary 7.2.10, there exists a Stein open neighbourhood U_2 of $K_{-\infty,a}$ and $f_2 = (f_2^1, \ldots, f_2^q) \in F(U_2)^q$ such that $f_2^1, \ldots, f_2^q$ generate F as an O-module over U_2. By Corollary 7.2.10, 2, we may find an open Stein neighbourhood U_3 of $K_{a,a}$ and $q \times p$ matrix θ_1, with coefficients holomorphic on U_3, such that

$$\theta_1 f_1 = f_2 \text{ on } U_3.$$

Similarly, we may find a $p \times q$ matrix θ_2, with coefficients holomorphic on some Stein open neighbourhood U_4 of $K_{a,a}$, such that

$$\theta_2 f_2 = f_1 \text{ on } U_4.$$

Shrinking $U_1,\ldots,U_4$ we may suppose that $U_1 \cap U_2 = U_3 = U_4$ (though, of course, the neighbourhoods need no longer be Stein). Since $K_{-\infty,b}$ is $A(\Omega)$-convex and contained in $U_1 \cup U_2$, there exists a Stein open neighbourhood U of $K_{-\infty,b}$ which is contained in $U_1 \cup U_2$ (Lemma 7.2.8). Replacing the neighbourhoods U_j by their intersections with U, we may assume that $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = U_3 = U_4$.

For $j = 1,2$, we define homomorphisms $F_j\colon \mathcal{O}^{p+q}_{U_j} \to \mathcal{F}_{U_j}$ by

$$F_1(u^1_1,\ldots,u^1_{p+q}) = \sum_{i=1}^{p} u^1_i f^i_1 \quad , \quad u^1 = (u^1_1,\ldots,u^1_{p+q}) \in A(U_1)^{p+q}$$

$$F_2(u^2_1,\ldots,u^2_{p+q}) = \sum_{i=1}^{q} u^2_{i+p} f^i_2; \quad u^2 = (u^2_1,\ldots,u^2_{p+q}) \in A(U_2)^{p+q}.$$

Our construction guarantees that the sequences

$$\mathcal{O}^{p+q}_{U_j} \xrightarrow{F_j} \mathcal{F}_{U_j} \to 0$$

are exact, $j = 1,2$. In the remainder of the proof we show how to identify the sequences over U_{12} to obtain a locally free resolution of $\mathcal{F}$ over U.

Let I_p, I_q denote the identity $p \times q$, $q \times q$ matrices respectively. We see that

$$\begin{pmatrix} I_p & 0 \\ \theta_1 & I_q \end{pmatrix}\begin{pmatrix} f_1 \\ 0 \end{pmatrix} = \begin{pmatrix} I_p & \theta_2 \\ 0 & I_q \end{pmatrix}\begin{pmatrix} 0 \\ f_2 \end{pmatrix} \quad \text{on } U_{12}.$$

Hence

$$\begin{pmatrix} f_1 \\ 0 \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ -\theta_1 & I_q \end{pmatrix}\begin{pmatrix} I_p & \theta_2 \\ 0 & I_q \end{pmatrix}\begin{pmatrix} 0 \\ f_2 \end{pmatrix} .$$

Let $\theta: U_{12} \to GL(p+q,\mathbb{C})$ denote the holomorphic matrix function defined by

$$\theta = \begin{pmatrix} I_p & 0 \\ -\theta_1 & I_q \end{pmatrix}\begin{pmatrix} I_p & \theta_2 \\ 0 & I_q \end{pmatrix}.$$

Observe that, over U_{12}, we have $F_1' = \theta F_2'$, where the prime denotes transpose. Therefore, $F_1 = F_2\theta'$. Hence, if $u^j \in A(U_j)^{p+q}$, $j=1,2$, $F_1u^1 = F_2u^2$ if and only if $F_2(\theta'u^1) = F_2u^2$. In particular, we have equality if $u^2 = \theta'u^1$. Setting $\phi_{12} = (\theta')^{-1}$, $\phi_{21} = \phi_{12}^{-1}$ we see that $\{\phi_{12},\phi_{21}\}$ are the transition functions for a holomorphic vector bundle E over $U = U_1 \cup U_2$ and that the condition $\phi_{21}u^1 = u^2$ amounts to saying that u^1, u^2 are local representatives of a holomorphic section u of E (over U). The morphisms F_1, F_2 determine a homomorphism $\underline{E} \to \mathcal{F}_U$ which is surjective since both F_1 and F_2 are surjective. We now apply Proposition 7.2.5, 2, to deduce that $K_{-\infty,b}$ possesses property (R). Contradiction. Therefore $S = +\infty$ and K possesses property (R). □

Lemma 7.2.12. Every compact $A(\Omega)$-convex subset K of a domain of holomorphy Ω in $\mathbb{C}^n$ possesses property (R).

Proof. The set $\{z \in K: \mathrm{Re}(z_j) = a_j, \mathrm{Im}(z_j) = b_j, j=1,\dots,n\}$ possesses property (R) for arbitrary $a_j,b_j \in \mathbb{R}$ since it either is empty or consists of a single point. By Lemma 7.2.11, the set obtained by dropping one of the conditions $\mathrm{Re}(z_j) = a_j$, $\mathrm{Im}(z_j) = b_j$ still has property (R). Iteration of this argument 2n times gives the result. □

The first part of the next lemma will enable us easily to extend Lemma 7.2.12 to an arbitrary Stein manifold. We shall use the second part to prove a key approximation theorem needed in the proof of Theorem B.

Lemma 7.2.13. Let K be a compact A(M)-convex subset of the Stein manifold M. Then, given an open neighbourhood U of K, we may find a Stein open neighbourhood $V \subset U$ of K and holomorphic map $F: M \to \mathbb{C}^N$ such that

1. F maps V biholomorphically onto a closed submanifold P of the unit Euclidean disc $E = \{(z_1,\ldots,z_N): \sum_{i=1}^{N} |z_i|^2\}$ of $\mathbb{C}^N$.

2. For $p \geq 1$, $H^p(E,I_P) = 0$ (I_P denotes the ideal sheaf of P).

Proof. (Rossi [1]). Without loss of generality we may suppose that U is relatively compact. For each $x \in \bar{U}$, we may find $f_1,\ldots,f_m \in A(M)$ which define local coordinates at x. Hence, by the compactness of $\bar{U}$, we may find a finite set $h_1,\ldots,h_p \in A(M)$ which give local coordinates at every point $x \in \bar{U}$. In particular, we may find an open neighbourhood W of the diagonal in $\bar{U} \times \bar{U}$ such that if $x,y \in W$, $x \neq y$, there exists an h_j with $h_j(x) \neq h_j(y)$. For each $x,y \in \bar{U} \times \bar{U} \setminus W$, we can find $f \in A(M)$ with $f(x) \neq f(y)$. Therefore since $\bar{U} \times \bar{U} \setminus W$ is compact we may find a finite set $h_{p+1},\ldots,h_r \in A(M)$ which separates points in $\bar{U} \times \bar{U} \setminus W$. We see at once that the set $h_1,\ldots,h_r$ separates points of $\bar{U}$ and gives local coordinates at every point of $\bar{U}$. In particular, the map $h = (h_1,\ldots,h_r): M \to \mathbb{C}^r$ restricts to an injective holomorphic immersion on some neighbourhood of $\bar{U}$. Multiplying h by a sufficiently small scalar we may in addition suppose that

$$\|h\|_K^2 = \sum_{j=1}^{r} \|h_j\|_K^2 < \tfrac{1}{2}.$$

Just as in the proof of Lemma 7.2.8, we may find $f_1,\ldots,f_k \in A(M)$ such that $\|f_j\|_K < 1$, $1 \leq j \leq k$ and $\max_j |f_j(z)| > 1$, all $z \in \partial U$. For positive integers q, define $g_q: M \to \mathbb{R}$ by

$$g_q(z) = \sum_{j=1}^{k} |f_j^q|^2.$$

Fix a value of q so large that $\|g_q\|_K < \tfrac{1}{2}$ and $|g_q(z)| > 1$, $z \in \partial U$. For $z \in M$, define

$$G(z) = g_q(z) + \sum_{j=1}^{r} |h_j(z)|^2$$

and let $V = \{z \in U: G(z) < 1\}$. Our construction guarantees that V is an open neighbourhood of K with $\bar{V} \subset U$. The map $F: M \to \mathbb{C}^{r+k}$ defined by $F = (h_1,\ldots,h_r,f_1^q,\ldots,f_k^q)$, maps V biholomorphically onto a closed submanifold P of E. In particular P is Stein, since E is Stein, and so V is Stein.

To complete the proof, notice that for ε small and positive, $V = \{z \in U: G(z) < 1+\varepsilon\}$ will be mapped biholomorphically by F onto a closed submanifold P_ε of the Euclidean disc E_ε of radius $1+\varepsilon$. Now $\mathcal{I}_{P_\varepsilon}|E = \mathcal{I}_P$ and so, since $\bar{E}$ is an $A(E_\varepsilon)$-convex compact subset of E_ε, Lemma 7.2.12 and Corollary 7.2.10 imply that $H^p(E,\mathcal{I}_P) = 0$, $p \geq 1$. □

Proof of Theorem 7.2.6. Let ω be a relatively compact Stein open subset of M. Then $K = \hat{\omega}$ is a compact A(M)-convex subset of M. By Lemma 7.2.13, we may find a Stein open neighbourhood V of K and holomorphic map F of M into some $\mathbb{C}^N$ such that F maps V biholomorphically onto a closed submanifold P of the unit Euclidean disc E in $\mathbb{C}^N$. Since F maps V biholomorphically onto P, $F_*\mathcal{F}_V$ is a coherent sheaf of $F_*\mathcal{O}_V = \mathcal{O}_P$-modules on P. Let $\tilde{\mathcal{F}}$ denote the trivial extension of $F_*\mathcal{F}_V$ to E. $\tilde{\mathcal{F}}$ is a coherent sheaf of $\mathcal{O}_E$-modules. Choose $\varepsilon > 0$ so that $(F|V)^{-1}E_\varepsilon \supset K$, where E_ε denotes the open Euclidean disc of radius $1-\varepsilon$ in $\mathbb{C}^N$. By Lemmas 7.2.9, 7.2.11, we may find a free resolution $0 \to \mathcal{O}_{E_\varepsilon}^{p_m} \to \dots \to \mathcal{O}_{E_\varepsilon}^{p_0} \to \tilde{\mathcal{F}}_{E_\varepsilon} \to 0$ of $\tilde{\mathcal{F}}$ over E_ε. Restrict the resolution to $P_\varepsilon = E_\varepsilon \cap P$ and apply Exercise 9, §1 to obtain a free $\mathcal{O}_P$-resolution of $F_*\mathcal{F}$ over P_ε. Pull back by F to obtain a free $\mathcal{O}_M$-resolution of $\mathcal{F}$ over $(F|V)^{-1}(E_\varepsilon) \supset K$. We have shown that $\mathcal{F}$ has a free resolution over some open neighbourhood of K and, a fortiori, over ω. □

As an immediate corollary of Theorem 7.2.6 we have

Lemma 7.2.14. Every compact A(M)-convex subset K of a Stein manifold M possesses property (R).

Proposition 7.2.15. Let $\mathcal{F}$ be a coherent sheaf on the Stein manifold M. Then $H^q(M,\mathcal{F}) = 0$, $q \geq 2$.

Proof. Using Lemma 7.2.8 we may choose an open cover $\mathcal{U} = \{U_i: i = 1,2,\dots\}$ of M by relatively compact Stein open subsets of M satisfying

1. $\bar{U}_n \subset U_{n+1}$, $n \geq 1$; 2. $\bigcup_{n=1}^{\infty} U_n = M$.

By Example 2 and Theorem 7.2.6, $\mathcal{U}$ is a Leray cover of M for $\mathcal{F}$. Hence $H^q(M,\mathcal{F}) \approx H^q(\mathcal{U},\mathcal{F})$, $q \geq 0$. Suppose $A \in Z^q(\mathcal{U},\mathcal{F})$. For $n \geq 1$, let $\mathcal{U}_n = \{U_j: j = 1,\dots,n\}$ and A_n denote the restriction of A to $\mathcal{U}_n$. Since $\mathcal{U}_n$ is a Leray cover of U_n for $\mathcal{F}$ and $H^q(U_n,\mathcal{F}) = 0$, $q \geq 1$, we have $H^q(U_n,F) = 0$, $q \geq 1$. Now $A_n \in Z^q(U_n,F)$ and so there exists $b_n \in C^{q-1}(U_n,F)$ such that $D(b_n) = A_n$. Extend b_n to $C^{q-1}(U,F)$ by setting $b_n|U_j = 0$, $j > n$. We construct inductively a sequence $B_n \in C^{q-1}(U,F)$ satisfying

1. $B_n|U_j = B_j|U_j$, $j \leq n$; 2. $A_n = D(B_n)|U_n$.

Define $B_1 = b_1$ and suppose $B_1,\ldots,B_n$ have been constructed. We have $D((b_{n+1} - B_n)|U_n) = 0$. So, provided that $q - 1 \geq 1$, there exists $C_n \in C^{q-2}(U_n,F)$ such that $D(C_n) = (b_{n+1} - B_n)|U_n$. Extend C_n to $C^{q-2}(U,F)$ by taking $C_n|U_j = 0$, $j > n$. We define $B_{n+1} = b_{n+1} - D(C_n)$. Finally define $B \in C^{q-1}(U,F)$ by $B|U_j = B_j$. Clearly $D(B) = A$. □

The proof of Proposition 7.2.15 clearly breaks down if $q = 1$ and we have to use an approximation theorem for this case. First we need to topologise the space of sections of a coherent sheaf.

Let K be a compact $A(M)$-convex subset of the Stein manifold M and F be a coherent sheaf on M. By Proposition 7.2.4 and Theorem 7.2.6 we may find sections $s_1,\ldots,s_k \in F(K)$ which generate F as a $\mathcal{O}$-module over K (see also Lemma 6.1.12). Let $s \in F(K)$. Define

$$|s|_K = \inf\left\{\max_j \|c_j\|_K : s = \sum_{j=1}^{k} c_j s_j,\ c_j \in \mathcal{O}(K)\right\}.$$

The seminorm $|\ |_K$ may depend on the choice of generators $s_1,\ldots,s_k$ but another choice of generators gives an equivalent seminorm since the two sets of generators are related by a matrix with holomorphic entries analytic on K, Corollary 7.2.10.

Lemma 7.2.16. Let $s \in F(K)$ and suppose that $|s|_K = 0$. Then $s_z = 0$, for all $z \in \mathring{K}$.

Proof. Choose generators $s_1,\ldots,s_k \in F(K)$ for F over K and suppose $s = \sum_{j=1}^{k} c_j s_j$ on K. If $|s|_K = 0$, we may find for every $\varepsilon > 0$, $c_j^\varepsilon \in \mathcal{O}(K)$ such that $s = \sum_{j=1}^{k} c_j^\varepsilon s_j$ and $\|c_j^\varepsilon\|_K < \varepsilon$, $1 \leq j \leq k$. We certainly have

$$(c_1 - c_1^\varepsilon,\ldots,c_k - c_k^\varepsilon)_z \in R(s_1,\ldots,s_k)_z,\ z \in K \quad \ldots.(*)$$

Fix $z \in \mathring{K}$ and let $P_1,\ldots,P_q \in R(s_1,\ldots,s_k)_z$ be a set of generators over $\mathcal{O}_z$ for $R(s_1,\ldots,s_k)_z$. By Theorem 3.6.2 and the Oka Theorem

(Theorem 7.1.6), there exists an open neighbourhood $D \subset \overset{\circ}{K}$ of z such that the sequence

$$A_B(D)^q \xrightarrow{P} R_B(s_1,\dots,s_k)(D) \to 0$$

is *split* exact ("B" denotes bounded sections). In particular, letting $\varepsilon \to 0$ in (*), we see that $(c_1,\dots,c_k)|D \in \mathrm{Im}(P) = R(s_1,\dots,s_k)(D)$. But therefore $s_y = 0$, $y \in D$. Hence the result. Of course, our argument is just the closure of modules theorem, Exercise 3, §6, Chapter 3. □

Let $(K_n)_{n\geq 1}$ be a normal exhaustion of M by compact A(M)-convex subsets K_n (see §4, Chapter 2 and note that we require $K_n \subset \overset{\circ}{K}_{n+1}$, $n \geq 1$). For each n choose sections in $F(K_n)$ which generate F over K_n and let $|\ |_n = |\ |_{K_n}$ denote the corresponding seminorm. We take the topology on $H^0(M,F)$ defined by the seminorms $|\ |_n$, $n \geq 1$. It is quite straightforward to verify that any two normal exhaustions of M will give rise to the same topology on $H^0(M,F)$.

Lemma 7.2.17. Suppose that we are given sections $f_n \in F(K_n)$, $n \geq 1$, such that for $p \geq 1$, $|f_n - f_m|_p \to 0$, $n,m \to \infty$. Then there exists a unique section $F \in F(M)$ such that $|F - f_n|_p \to 0$, $n \to \infty$, $p \geq 1$.

Proof. Fix $p \geq 1$ and let $s_1,\dots,s_k \in F(K_{p+1})$ be sections defining $|\ |_{p+1}$. Choose integers $n_1 \leq n_2 \leq \dots$ such that

$$\left|f_{n_{i+1}} - f_{n_i}\right|_{p+1} \leq 2^{-i},\ i \geq 1.$$

Set $u_0 = f_{n_1}$, $u_i = f_{n_{i+1}} - f_{n_i}$, $i \geq 1$. Choose $c_{ij} \in O(K_{p+1})$ such that $u_i = \sum_{j=1}^{k} c_{ij}s_j$ and $\max_j \|c_{ij}\|_{K_{p+1}} \leq |u_i|_{p+1} + 2^{-i} \leq 2^{-i+1}$. Since $\sum_{i=1}^{\infty} \|c_{ij}\|_{K_{p+1}} < \infty$, the series $\sum_{i=0}^{\infty} c_{ij}$ converges uniformly on K_{p+1} to a function C_j which is analytic on $\overset{\circ}{K}_{p+1}$, $1 \leq j \leq k$. Since $K_p \subset \overset{\circ}{K}_{p+1}$, $C_j \in O(K_p)$. Define $F_p = \sum_{j=1}^{k} C_j s_j$ and observe that

$$\left|F_p - f_{n_{q+1}}\right|_p = \left|F_p - \sum_{i=0}^{q} u_i\right|_p \to 0 \text{ as } q \to \infty.$$

Now by the triangle-inequality it is clear that $|F_p - f_n|_p \to 0$, $n \to \infty$. Moreover by Lemma 7.2.16, this condition determines F_p uniquely on $\mathring{K}_p$. In particular, $F_{p+r}|\mathring{K}_p = F_p|\mathring{K}_p$, $r \geq 0$. Define $F \in F(M)$ by $F|\mathring{K}_p = F_p$. □

Theorem 7.2.18. Let F be a coherent sheaf on the Stein manifold M and give $H^0(M,F)$ the topology described above. Then $H^0(M,F)$ is a Fréchet space. □

Proof. Lemmas 7.2.16, 7.2.17. □

Next we prove our main approximation theorem.

Theorem 7.2.19. Let K be an A(M)-convex subset of the Stein manifold M and suppose that $f \in O(K)$. Then there exists a sequence $f_n \in A(M)$ such that $\|f_n - f\|_K \to 0$, $n \to \infty$.

Proof. We may suppose that $f \in A(U)$, where U is an open neighbourhood of K. By Lemma 7.2.13, we may find a Stein open neighbourhood $V \subset U$ of K and analytic map $F: M \to \mathbb{C}^N$ such that $P = F(V)$ is a closed submanifold of the unit Euclidean disc E in $\mathbb{C}^N$ and $H^p(E, I_P) = 0$, $p \geq 1$. Let $g = f(F|V)^{-1} \in A(P)$. Taking the cohomology sequence of $0 \to I_P \to O_E \to O_P \to 0$ we see that the restriction map $A(E) \to A(P)$ is onto. Therefore there exists $G \in A(E)$ such that $G|P = g$. Now polynomials are dense in A(E) (Exercise 1, §2, Chapter 2) Therefore, for $n \geq 1$ we may find a polynomial p_n on $\mathbb{C}^N$ such that $\|p_n - G\|_{F(K)} \leq 1/n$. Set $f_n = p_n F \in A(M)$. Then $\|f_n - f\|_K \to 0$, $n \to \infty$. □

Theorem 7.2.20. Let K be a compact A(M)-convex subset of the Stein manifold M and F be a coherent sheaf on M. Suppose $f \in F(K)$. Then there exists a sequence $f_n \in F(M)$ such that $|f - f_n|_K \to 0$, $n \to \infty$.

Proof. Choose a normal exhaustion (K_p) of M with $K_1 = K$. Fix $n \geq 1$. For $p > 0$, we shall construct $g_p \in F(K_p)$ such that

$$g_1 = f \text{ and } |g_p - g_{p+1}|_r \leq 2^{-p}/n, \ r \leq p.$$

By Lemma 7.2.17 this will imply that there exists $f_n \in F(M)$ such that $|f_n - g_j|_p \to 0$, $j \to \infty$, $p \geq 1$. Taking $p = 1$ and noting that

$f_n = \sum_{j=1}^{\infty} (g_{j+1} - g_j) + f$ on K, we therefore have $|f_n - f|_K \leq 1/n$.

To construct the sections g_p we proceed inductively and suppose $g_1,\ldots,g_p$ constructed. Let $s_1,\ldots,s_k \in F(K_{p+1})$ generate F over K_{p+1}. Then $g_p = \sum_{j=1}^{k} c_j s_j$, $c_j \in O(K_p)$. By Theorem 7.2.19, we may find $d_j \in A(M)$ such that $\|c_j - d_j\|_{K_p} \le 2^{-p}/n$. Clearly $g_{p+1} = \sum_{j=1}^{k} d_j s_j$ satisfies our requirements. □

Theorem 7.2.21. (**Theorem B of Cartan**). Let F be a coherent sheaf on the Stein manifold M. Then $H^q(M,F) = 0$, $q \ge 1$.

Proof. We have already proved that $H^q(M,F) = 0$, $q \ge 2$ (Proposition 7.2.15). There remains the case $q = 1$.

Choose a normal exhaustion $(K_n)_{n\ge1}$ of M. For each n we may, by Lemma 7.2.8, find a relatively compact Stein open neighbourhood $U_n \subset \mathring{K}_{n+1}$ of K_n. Certainly $U_n \subset U_{n+1}$, $n \ge 1$. As in the proof of Proposition 7.2.15, $U = \{U_n\}$ is a Leray cover of M for F. Set $U_n = \{U_i: i = 1,\ldots,n\}$, $n \ge 1$. Given $A \in Z^1(U,F)$, let A_n denote the restriction of A to U_n. Since $H^1(U_n,F) = 0$, there exists $b_n \in C^0(U_n,F)$ such that $D(b_n) = A_n$, $n \ge 1$. We construct inductively a sequence $B_n \in C^0(U_n,F)$ such that

1. $D(B_n) = A_n$.

2. $|B_n - B_{n+1}|_r \le 2^{-n}$, $1 \le r \le n$.

(For 2 note that 1 implies $B_n - B_{n+1} \in F(U_n)$). Take $B_1 = b_1$ and suppose $B_1,\ldots,B_n$ have been constructed. Condition 1 implies $b_{n+1} - B_n \in F(U_n)$ and so by Theorem 7.2.20 there exists $\eta \in F(M)$ such that $|B_n - b_{n+1} - \eta|_n \le 2^{-n}$. Now take $B_{n+1} = b_{n+1} + \eta$. Next we shall construct $F_p \in F(\mathring{K}_p)$ such that $B_p + F_p = B_{p+1} + F_{p+1}$ on $\mathring{K}_p$, $p \ge 1$. Given $p \ge 1$, let $s_1,\ldots,s_k \in F(K_p)$ define the seminorm $|\ |_p$. We may choose $c_{ij} \in O(K_p)$ such that for $i \ge p$, $(B_{i+1} - B_i)|K_p = \sum_{j=1}^{k} c_{ij}s_j$ and $\max_j\|c_{ij}\|_{K_p} \le 2^{-i+1}$. Just as in the proof of Lemma 7.2.17 these conditions imply that for $1 \le j \le k$, $\sum_{i=p}^{\infty} c_{ij}$ converges uniformly on K_p to a function C_j which is analytic on $\mathring{K}_p$. Set $F_p = \sum_{j=1}^{k} C_j s_j \in F(\mathring{K}_p)$. For every A(M)-convex compact subset K of $\mathring{K}_p$ we have

$$\lim_{n\to\infty}|(F_p + B_p) - B_{n+1}|_K = \lim_{n\to\infty}|F_p - \sum_{i=p}^{n}(B_{i+1} - B_i)|_K = 0.$$

It follows from Lemma 7.2.16 that $B_p + F_p = B_{p+1} + F_{p+1}$ on $\mathring{K}_p$. Hence we may define $B \in C^0(\mathcal{U},F)$ by $B|\mathring{K}_p = B_p + F_p$, $p \geq 1$. Now as $D(B_{n+1})|U_n = A_n$ and F_{n+1} restricts to a section of F over U_n - that is a cocycle - we see that $D(B)|U_n = A_n$, $n \geq 1$. Hence $D(B) = A$. □

Theorem 7.2.22 (Theorem A of Cartan). Let F be a coherent sheaf on the Stein manifold M. For each $x \in M$, there exists $s_1,\ldots,s_p \in F(M)$ such that $s_{1,x},\ldots,s_{p,x}$ generate F_x as an $\mathcal{O}_x$-module.

Proof. Let I denote the ideal sheaf of $\{x\}$. Then IF is a coherent subsheaf of F, Theorem 7.1.9, 1. Therefore by Theorem B, $H^1(M,IF) = 0$ and so taking the cohomology sequence of $0 \to IF \to F \to F|IF \to 0$ we obtain the exact sequence

$$F(M) \to (F/IF)(M) \to 0 \qquad \ldots.(*)$$

For $y \neq x$, $(IF)_y = F_y$ and so $(F/IF)_y = 0$. Hence $(F/IF)(M) = F_x/m_xF_x$, where m_x denotes the maximal ideal of $\mathcal{O}_x$ at x. Let N denote the $\mathcal{O}_x$-submodule of F_x generated by $\{s_x : s \in F(M)\}$. Since (*) is exact we have $N + m_xF_x = F_x$. It now follows by Nakayama's Lemma that $N = F_x$. □

We may now give a global version of Corollary 7.2.10.

Theorem 7.2.23. Let F be a coherent sheaf on the Stein manifold M. Suppose that $s_1,\ldots,s_p \in F(M)$ generate F_x for every $x \in M$. Then given $S \in F(M)$, there exist $f_j \in A(M)$ such that

$$S = \sum_{j=1}^{p} f_j s_j.$$

Proof. The sheaf map $s = (s_1,\ldots,s_p): \mathcal{O}_M^p \to F \to 0$ is onto. Since $\mathrm{Ker}(s)$ is coherent, Theorem 7.1.2, $H^1(M,\mathrm{Ker}(s)) = 0$ by Theorem B. Hence taking the cohomology sequence of $0 \to \mathrm{Ker}(s) \to \mathcal{O}_M^p \xrightarrow{s} F \to 0$ we obtain the exact sequence $A(M)^p \xrightarrow{s} F(M) \to 0$. □

Remarks. Our proof of Theorem B is close to that of Hörmander [1] in that it makes use of Theorem B for locally free sheaves of $\mathcal{O}$-modules. The main difference is that we make use of a device of Rossi [1] to give an elementary proof of the approximation theorem, Theorem 7.2.19. Cartan's original proof starts by establishing Theorems A and B for cubes in $\mathbb{C}^n$ and then extends the result to Stein manifolds. For expositions of this approach to Theorems A and B see H. Cartan [1], Grauert and Remmert [1] and Gunning and Rossi [1]. Rossi [1] gives a proof of Theorems A and B by starting from the relatively elementary result of Oka to the effect that $H^q(D,\mathcal{O}_D) = 0$, $q \geq 1$, for all polynomially convex domains D in $\mathbb{C}^n$. Using this result he constructs arbitrarily fine Leray covers for coherent sheaves and then proves Grauert's finiteness theorem for strictly pseudoconvex domains (see §4). Rossi then shows that the cohomology of a coherent sheaf on a strictly pseudoconvex domain is supported on the compact analytic subsets of the domain (see also Rossi [2], R. Narasimhan [3]). Since Stein manifolds have no non-trivial compact analytic subsets this is sufficient to deduce the vanishing theroem for relatively compact strictly pseudoconvex domains in a Stein manifold. The rest of his proof is similar to what we have presented here.

One point about our proof should be noticed: We really only assume Theorem B for locally free sheaves defined over *contractible* domains in $\mathbb{C}^n$. In fact by a theorem of Grauert, such locally free sheaves are free. Of course if we knew this, we could apply an appropriate version of the (elementary) Dolbeault-Grothendieck lemma to prove Theorems A and B. However, a direct proof that locally free sheaves over contractible domains in $\mathbb{C}^n$ are free is not easy. It is a main step in the original proof of H. Cartan (and then only for a restricted class of contractible domains in $\mathbb{C}^n$). A proof of the triviality of locally free sheaves over contractible domains in $\mathbb{C}^n$ which uses Theorem B for locally free sheaves may be found in Adams and Griffiths [1].

One merit of the partial differential equation techniques used in establishing the exactness of the $\bar{\partial}$-seqeunce on a Stein manifold is that they give good estimates on the growth of solutions to $\bar{\partial}f = g$. The resulting "cohomology theory with bounds" has important applications to the theory of partial differential equations and is described in Hörmander [1].

Exercises.

1. Show that Theorem 7.2.20 is false if K is not A(M)-convex.

2. Prove that if M is a complex manifold such that $H^1(M,\mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$ of $\mathcal{O}_M$ then M is Stein (Hint: For the holomorphic convexivity of M show that given any discrete subset $\{x_i : i \geq 1\}$ of M there exists $f \in A(M)$ such that $f(x_i) = i$, $i \geq 1$. See also Seminar number 20 by J. Serre in H. Cartan [2]).

3. Show that if M is an m-dimensional Stein manifold then $H^r(M,\mathbb{C}) = 0$, $r > m$ (Use Example 23, §1, Chapter 6).

4. Let M be a Stein manifold and suppose $H^2(M,\mathbb{Z}) = 0$. Show that any non-zero meromorphic function on M may be written in the form f/g, $f,g \in A(M)$ and $(f_x,g_x) = 1$, $x \in M$.

5. Let M be a Stein manifold and suppose m is a meromorphic function on M. Show that there exist $f,g \in A(M)$ such that $m = f/g$ (Hint: for $z \in M$, let $\hat{F}_z$ denote the ideal of $\mathcal{O}_z$ consisting of all germs g_z such that $g_z m_z \in F_z$. Prove that the sheaf $\hat{F}$ is coherent and find a non-trivial section of $\hat{F}$). In Chapter 12 we give examples to show that we cannot generally require $(f_z,g_z) = 1$, $z \in M$.

6. Let M be a Stein manifold with $H^2(M,\mathbb{Z}) = 0$ and suppose that H is an analytic hypersurface in M. Prove that $M \setminus H$ is Stein.

7. Suppose that $\{x_i\}$ is a discrete subset of the Stein manifold M and that for each i we are given a Laurent series

$$L_i = \sum_{m=-N(i)}^{P(i)} a_m(z-x_i)^m, \quad 0 \leq P(i),\ N(i) < \infty.$$

Show that there exists a meromorphic function m on M such that for all i, $(m - L_i)$ is holomorphic on some neighbourhood of x_i and has a zero of order $P(i)+1$ at z_i (see Exercise, §3, Chapter 1).

8. Let H be an analytic hypersurface in the Stein manifold M. Show that $\mathcal{O}_H$ has a resolution by locally free sheaves of the form

$$0 \to \underline{[H]}^{-1} \to \mathcal{O}_M \to \mathcal{O}_H \to 0 .$$

9. Show that a coherent sheaf on $\mathbb{C}^n$, $n > 1$, need not have a resolution by (locally) free sheaves of $\mathcal{O}_{\mathbb{C}^n}$-modules (Hint: Given $z \in \mathbb{C}^n$ and $p \geq 1$, consider resolutions of the coherent sheaf $\mathcal{O}_{\mathbb{C}^n}/I^p_{\{z\}}$).

§3. The finiteness theorem of Cartan and Serre.

The aim of this section is to prove that $\dim H^q(M,\mathcal{F}) < \infty$, $q \geq 0$, for all coherent sheaves $\mathcal{F}$ on a compact complex manifold M. We start by reviewing some definitions and results about Fréchet spaces. A general reference for Fréchet spaces is Rudin [1]. See also Appendix B in Gunning and Rossi [1] .

Lemma 7.3.1. Finite direct sums and countable products of Fréchet spaces are Fréchet. If G is a closed subspace of the Fréchet space E then G and E/G are Fréchet.

Proof. We shall prove that E/G is Fréchet if G is a closed subspace of E and leave the remaining assertions as elementary exercises for the reader. Suppose that the topology on E is defined by the seminorms $|\ |_p$, $p \geq 1$. Let $q: E \to E/G$ denote the quotient map. For $p \geq 1$, define

$$|q(u)|'_p = \inf_{g \in G}|u-g|_p, \quad u \in E.$$

Then $|\ |'_p$ are seminorms on E/G defining the quotient topology on E/G. Since G is a closed subspace of E, E/G is Hausdorff and it is straightforward to verify that the seminorms $|\ |'_p$ define the structure of a Fréchet space on E/G. □

Theorem 7.3.2. (Open mapping theorem). Let $A: E \to F$ be a continuous surjective linear map between Fréchet spaces. Then A is open. In particular, if A is a continuous linear bijection, A is a homeomorphism.

Proof. Rudin [1; Corollary 2.12], Gunning and Rossi [1; Appendix B]. □

Theorem 7.3.3. A locally compact Fréchet space is finite dimensional.

Proof. Rudin [1; Theorem 1.22], Gunning and Rossi [1; Appendix B]. □

Definition 7.3.4. Let $A: E \to F$ be a continuous linear map between Fréchet spaces. We say A is *compact* if there exists an open neighbourhood V of 0 in E such that A(V) is relatively compact.

Theorem 7.3.5. (L. Schwartz). Let $A,B: E \to F$ be continuous linear maps between Fréchet spaces and suppose that A is surjective and B is compact. Then $\mathrm{Im}(A+B)$ is a closed subspace of F and $F/\mathrm{Im}(A+B)$ is finite dimensional.

Proof. Gunning and Rossi [1; Appendix B]. □

Suppose that U is an open subset of the complex manifold M. We may give $\mathcal{O}(U)$ the structure of a Fréchet space by taking as seminorms

$$|f|_p = \|f\|_{K_p}, \quad f \in \mathcal{O}(U),$$

where $(K_p)_{p\geq 1}$ is any increasing family of compact subsets of U satisfying $\cup K_p = U$ and $K_p \subset \mathring{K}_{p+1}$, $p \geq 1$. Indeed, the topology defined by the seminorms is just the topology of uniform convergence on compact subsets of U. In particular, it is independent of the choice of sequence $(K_p)_{p\geq 1}$ and corresponding seminorms. It follows immediately from Lemma 7.3.1, that $\mathcal{O}(U)^p$ has the structure of a Fréchet space, $p \geq 1$.

Now suppose K is a coherent subsheaf of $\mathcal{O}_U^p$. Just as in the proof of Lemma 7.2.16, the closure of modules theorem implies that K(U) is a closed subspace of $\mathcal{O}(U)^p$. Hence, by Lemma 7.3.1, K(U) is Fréchet.

We shall say that an open relatively compact subset U of M is C-*admissible* if U is a Stein open subset of M and there exists another Stein open subset V of M such that $\bar{U} \subset V$.

Suppose that U is C-admissible and F is a coherent sheaf on M. By Theorem 7.2.6 and Proposition 7.2.4 there exist $s_1,\ldots,s_k \in F(U)$ which generate $F(U)$ as an $\mathcal{O}(U)$-module. Taking the cohomology sequence of the exact sequence $0 \to \mathcal{R}(s_1,\ldots,s_k) \to \mathcal{O}_U^k \to \mathcal{F}_U \to 0$ and applying Theorem B, we arrive at the exact sequence

$$0 \to \mathcal{R}(s_1,\ldots,s_k)(U) \to \mathcal{O}(U)^k \to \mathcal{F}(U) \to 0.$$

As noted above, $\mathcal{R}(s_1,\ldots,s_k)$ is a closed subspace of $\mathcal{O}(U)^k$ and so, by Lemma 7.3.1, $\mathcal{F}(U)$ has the structure of a Fréchet space.

We have already defined a Fréchet topology on $\mathcal{F}(U)$, Theorem 7.2.18. Noting the definition of the seminorms on the quotient $\mathcal{F}(U) = \mathcal{O}^k(U)/\mathcal{R}(s_1,\ldots,s_k)(U)$, it is clear that these two topologies coincide. In particular, the topology we have defined on $\mathcal{F}(U)$ is independent of the choice of generators $s_1,\ldots,s_k$ (see the discussion in §2).

From now on, assume that $\mathcal{F}(U)$ is topologised as a Fréchet space for all C-admissible subsets U of M.

Lemma 7.3.6. Let U, V be C-admissible subsets of M with $V \subset U$. Then the restriction map $r_{VU}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ is continuous.

Proof. Certainly, the restriction maps are continuous if $\mathcal{F} = \mathcal{O}^p$. Hence they are continuous if $\mathcal{F}$ is a coherent subsheaf of $\mathcal{O}_U^p$ and so, taking quotients, the result follows in general. □

Suppose now that U is an arbitrary open subset of M. Since M has a basis of open sets consisting of C-admissible sets (for example, biholomorphic images of polydiscs), we may write

$$U = \bigcup_{j=1}^{\infty} U_j,$$

where the U_j are C-admissible open subsets of M. By Lemma 7.3.1, $\prod_{j=1}^{\infty} \mathcal{F}(U_j)$ has the structure of a Fréchet space. Let $Z \subset \prod_{j=1}^{\infty} \mathcal{F}(U_j)$ be the subset defined by $Z = \{(f_j)\colon f_j = f_k \text{ on } U_{jk},\ j,k \geq 1\}$ and $K\colon \prod_{j=1}^{\infty} \mathcal{F}(U_j) \to \prod_{1\leq j<k<\infty} \mathcal{F}(U_{jk})$ be the linear map defined by $K((f_j)) = ((f_j - f_k)|U_{jk})$. By Lemma 7.3.6, K is continuous and therefore $Z = K^{-1}(0)$ is a closed subspace of $\prod_{j=1}^{\infty} \mathcal{F}(U_j)$. Hence by Lemma 7.3.1, Z is Fréchet. Now define $\chi\colon \mathcal{F}(U) \to \prod_{j=1}^{\infty} \mathcal{F}(U_j)$ by $\chi(f) = (f|U_j)$.

We see that χ maps $F(U)$ bijectively onto Z and so $F(U)$ inherits the Fréchet space structure of Z. Moreover, the Fréchet space structure we have defined on $F(U)$ is independent of the decomposition of U as a countable union of C-admissible sets. For suppose $U = \bigcup_{i=1}^{\infty} U_i^1 = \bigcup_{j=1}^{\infty} U_j^2$ are two such decompositions of U. Then $U = \bigcup_{i,j} U_i^1 \cap U_j^2$ is also a decomposition of U as C-admissible sets. Denote the corresponding closed subspaces of $\prod F(U_i^1)$, $\prod F(U_j^2)$, $\prod F(U_i^1 \cap U_j^2)$ by Z_1, Z_2, Z_{12} respectively. We have the commutative diagram

$$\begin{array}{ccccc} & & \prod F(U_i^1) & & \\ & \overset{\chi_1}{\nearrow} & & \overset{r_1}{\searrow} & \\ F(U) & & & & \prod F(U_i^1 \cap U_j^2) \\ & \underset{\chi_2}{\searrow} & & \underset{r_2}{\nearrow} & \\ & & \prod F(U_j^2) & & \end{array}$$

where r_1 and r_2 are induced by restriction and are therefore continuous by Lemma 7.3.6. Now r_1 and r_2 restrict to continuous bijections of Z_1 and Z_2 on Z_{12}. Therefore by the open mapping theorem (Theorem 7.3.2), Z_1 is homeomorphic to Z_{12} which in turn is homeomorphic to Z_2.

From now on assume that $F(U)$ is topologised as a Fréchet space according to the recipe above for all open subsets U of M.

Suppose that G is another coherent sheaf on M and $\phi: F \to G$ is a homomorphism. We claim that for all open subsets U of M the induced map $\phi_U: F(U) \to G(U)$ is continuous. By our definition of the topologies on $F(U)$, $G(U)$ it is clearly enough to verify that ϕ_U is continuous in case U is C-admissible. If U is C-admissible, there exist exact sequences $O(U)^p \xrightarrow{s} F(U) \to 0$, $O(U)^q \xrightarrow{t} G(U) \to 0$. Take the standard bases $\{E_1,\ldots,E_p\}$, $\{E_1,\ldots,E_q\}$ of $O(U)^p$, $O(U)^q$ (see §1). For $1 \le j \le p$, there exist $r_{ij} \in O(U)$ such that

$$\phi_U(s(E_j)) = \sum_{i=1}^{q} r_{ij} t(E_i).$$

Defining $\psi_U: O(U)^p \to O(U)^q$ by the matrix $[r_{ij}]$ we obtain the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(U)^p & \xrightarrow{s} & F(U) & \longrightarrow & 0 \\ \downarrow \psi_U & & \downarrow \phi_U & & \\ \mathcal{O}(U)^q & \xrightarrow{t} & G(U) & \longrightarrow & 0 . \end{array}$$

Now ψ_U is obviously continuous and so, since the quotient maps s and t are continuous, it follows from the Open Mapping theorem that ϕ_U is continuous.

Definition 7.3.7. A sheaf F of $\mathcal{O}$-modules on a complex manifold M is said to be a *Fréchet sheaf* if

a) For each open subset U of M, $F(U)$ is a Fréchet space.

b) The restriction maps r_{VU}: $F(U) \to F(V)$, $V \subset U$, are all continuous.

Theorem 7.3.8. Let M be a complex manifold. Then there is a unique way of giving every coherent sheaf defined over an open subset of M the structure of a Fréchet sheaf satisfying:

a) If U is an open subset of M and F is a coherent subsheaf of $\mathcal{O}_U^p$, then $F(U) \subset \mathcal{O}(U)^p$ has the topology of uniform convergence on compact subsets.

b) If ϕ: $F \to G$ is a homomorphism of coherent sheaves, defined over some open subset V of M, then the induced maps ϕ_U: $F(U) \to G(U)$ are continuous for every open subset $U \subset V$.

Proof. The existence part of the theorem follows from the discussion preceding Definition 7.3.7. For the uniqueness it is easily seen that it is enough to verify that conditions a) and b) determine the Fréchet space structure on $F(U)$ for U C-admissible. If U is C-admissible we have an exact sequence

$$0 \to R(U) \xrightarrow{s_U} \mathcal{O}^p(U) \xrightarrow{t_U} F(U) \to 0 .$$

Condition b) implies that s_U, t_U are continuous. Condition a) determines the topology on $R(U)$, $\mathcal{O}^p(U)$ and hence, by the open mapping theorem, the topology on $F(U)$. □

Remark. Our discussion of the Fréchet sheaf structure on coherent sheaves is based on Gunning and Rossi [1]. For much more extensive treatments of the topologisation of spaces of sections of coherent sheaves over complex manifolds and analytic sets see Gunning and Rossi [1] and Grauert and Remmert [1]. In the latter text a very nice characterisation of the Fréchet sheaf structure on coherent sheaves is given based on the Fréchet space structure on the stalks (see Exercise 4, §6, Chapter 3).

Lemma 7.3.9. Let U and V be open subsets of M and suppose $\bar{V} \subset U$, $\bar{V}$ compact. Then $r_{VU}: \mathcal{O}^p(U) \to \mathcal{O}^p(V)$ is compact.

Proof. Montel's theorem. □

Lemma 7.3.10. Let $\mathcal{F}$ be a coherent sheaf on M and suppose that U, V are open subsets of M with $\bar{V} \subset U$, $\bar{V}$ compact. Then $r_{VU}: \mathcal{F}(U) \to \mathcal{F}(V)$ is compact.

Proof. First suppose U is C-admissible. We have a commutative diagram of continuous maps

$$\begin{array}{ccccc} \mathcal{O}^p(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & 0 \\ \downarrow r_{VU} & & \downarrow r_{VU} & & \\ \mathcal{O}^p(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & 0 \end{array} .$$

Since $r_{VU}: \mathcal{O}^p(U) \to \mathcal{O}^p(V)$ is compact, Lemma 7.3.9, it follows that $r_{VU}: \mathcal{F}(U) \to \mathcal{F}(V)$ is compact.

For the general case, choose covers $\{U_j: j = 1,\dots,n\}$, $\{V_j: j = 1,\dots,n\}$ of V by C-admissible sets such that $\bar{V}_j \subset U_j$ is compact and $U_j \subset U$, $j = 1,\dots,n$. Then $r_{V_jU_j}: \mathcal{F}(U_j) \to \mathcal{F}(V_j)$ is compact and so therefore is

$$\prod_{j=1}^{n} r_{V_jU_j} : \prod_{j=1}^{n} \mathcal{F}(U_j) \to \prod_{j=1}^{n} \mathcal{F}(V_j) .$$

Set $\tilde{U} = \bigcup_{j=1}^{n} U_j$. The map $r_{V\tilde{U}}: \mathcal{F}(\tilde{U}) \to \mathcal{F}(V)$ factors through $\prod r_{V_jU_j}$ and so is compact. Hence $r_{VU} = r_{V\tilde{U}}r_{\tilde{U}U}$ is compact. □

Theorem 7.3.11. Let $\mathcal{F}$ be a coherent sheaf on the compact complex manifold M. Then $\dim_{\mathbb{C}} H^0(M,\mathcal{F}) < \infty$.

Proof. Take U = V = M in Lemma 7.3.10 and apply Theorem 7.3.3.

□

Theorem 7.3.12. (Cartan-Serre). Let $\mathcal{F}$ be a coherent sheaf on the compact complex manifold M. Then $\dim_{\mathbb{C}} H^q(M,\mathcal{F}) < \infty$, $q \geq 0$.

Proof. Choose Leray covers $\mathcal{U} = \{U_1,\ldots,U_n\}$, $\mathcal{U}' = \{U_1',\ldots,U_n'\}$ of M for $\mathcal{F}$ such that $\bar{U}_j' \subset U_j$, $j = 1,\ldots,n$. For $p \geq 0$, let

$$C^p = \bigoplus \mathcal{F}(U_s),$$

where the (finite) direct sum is taken over all distinct (p+1)-tuples $s = (s_0,\ldots,s_p)$ of integers satisfying $1 \leq s_0,\ldots,s_p \leq n$. By Lemma 7.3.1, C^p has the structure of a Fréchet space. Now $C^p(\mathcal{U},\mathcal{F})$ clearly defines a closed subspace of C^p (remember that $C^p(\mathcal{U},\mathcal{F})$ is the space of *alternating* cochains). Hence, $C^p(\mathcal{U},\mathcal{F})$ has the structure of a Fréchet space. Similarly, $C^p(\mathcal{U}',\mathcal{F})$ has the structure of a Fréchet space. Let $R: C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U}',\mathcal{F})$ denote the restriction homomorphism. By our assumptions on the covers $\mathcal{U}$, $\mathcal{U}'$ and Lemma 7.3.10, R is compact. Since the covers $\mathcal{U}$, $\mathcal{U}'$ are Leray for $\mathcal{F}$, the natural map

$$Z^p(\mathcal{U},\mathcal{F}) \to Z^p(\mathcal{U}',\mathcal{F})/B^p(\mathcal{U}',\mathcal{F})$$

is surjective and so

$$C^{p-1}(\mathcal{U}',\mathcal{F}) \oplus Z^p(\mathcal{U},\mathcal{F}) \xrightarrow{D \oplus R} Z^p(\mathcal{U}',\mathcal{F})$$

is surjective. But now take $A = D \oplus R$, $B = -0 \oplus R$ in Schwartz' finiteness theorem (Theorem 7.3.5) and we see that

$$H^p(M,\mathcal{F}) = H^p(\mathcal{U}',\mathcal{F}) = C^{p-1}(\mathcal{U}',\mathcal{F}) \oplus Z^p(\mathcal{U},\mathcal{F})/\mathrm{Im}(D)$$

is finite dimensional. □

Remarks.

1. We shall give an important application of the finiteness theorem in §5.

2. For generalisations of the finiteness theorem to coherent sheaves over compact analyitc spaces see Gunning and Rossi [1] and Grauert and Remmert [1] (the original theorem of Cartan and Serre was proved for compact analytic spaces).

Exercises.

1. Show that the finiteness theorem of Cartan-Serre is an immediate consequence of Grauert's direct image theorem (look at constant maps).

2*. (Gunning [2], Grauert and Remmert [1]). Let (U,ϕ) be a chart on the compact complex manifold M such that ϕ mapa U biholomorphically onto a polydisc in $\mathbb{C}^m$ ($m = \dim(M)$). Set $\mathcal{O}_h(U) = L^2(U)$ and note that $\mathcal{O}_h(U)$ has the structure of a Hilbert space (§6, Chapter 2; $\mathcal{O}_h(U)$ will depend on the chart map ϕ). Given a coherent sheaf F on M, show how to define the space $C_h^p(\mathcal{U},F)$ of square integrable cochains on $\mathcal{U}$, where $\mathcal{U}$ will be a cover of M by open polydiscs, and prove that $C_h^p(\mathcal{U},F)$ has the structure of a Hilbert space. Let $H_h^p(\mathcal{U},F)$ denote the corresponding cohomology group defined using square integrable cochains and prove that $\check{H}_h^p(M,F) \cong \check{H}^p(M,F)$, $p \geq 0$. Finally deduce the finiteness theorem of Cartan-Serre by using the elementary finiteness theorem of Schwartz for *Hilbert* spaces (we shall prove this finiteness theorem in the appendix to Chapter 10).

§4. The finiteness theorem of Grauert.

Suppose that M is a strictly Levi pseudoconvex (s.L.p) domain in $\hat{M}$ and that F is a coherent sheaf on $\hat{M}$. In this aection we shall prove the theorem of Grauert that the cohomology groups $H^p(M,F)$ are finite dimensional $\mathbb{C}$-vector apaces, $p \geq 1$. As in the proof of the finiteness theorem of Cartan-Serre, we shall make use of Schwartz' finiteness theorem. We use Grauert's finiteness theorem in §6 to give a proof of Kodaira's embedding theorem and again in Chapter 12 to construct real analytic embeddings.

Let $\| \; \|$ denote the standard Euclidean norm on $\mathbb{C}^n$ and E(r) denote the open Euclidean disc centre 0, radius r in $\mathbb{C}^n$. Given an open subset U of $\mathbb{C}^n$ we let $C^2_B(U)$ denote the space of C^2 $\mathbb{R}$-valued functions on U which, together with derivatives up to order 2, are bounded on U. Define a norm on $C^2_B(U)$ by

$$|\phi| = \sup_{x \in U}(\|\phi(x)\| + \|D^2\phi_x\|), \quad \phi \in C^2_B(U).$$

($\|D^2\phi_x\|$ denotes the polynomial norm of the bilinear map $D^2\phi_x$. That is, $\|D^2\phi_x\| = \sup_{\|v\|=1} \|D^2\phi_x(v^2)\|$ - see Dieudonné [1] or Field [1]). Recall from §10 of Chapter 5 that the Levi form $L(\phi)$ of a C^2 $\mathbb{R}$-valued map ϕ is the Hermitian quadratic form $\partial\bar{\partial}\phi$ given in local coordinates by the matrix $\left[\dfrac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\right]$.

Lemma 7.4.1. Let ϕ be a C^2 $\mathbb{R}$-valued function defined on some neighbourhood of 0 in $\mathbb{C}^n$. Suppose that for all $x \in \phi^{-1}(0)$, we have $d\phi(x) \neq 0$ and $L(\phi)(x)$ positive definite. Then there exists $r > 0$ and an open neighbourhood N of ϕ in $C^2_B(E(r))$ such that

1. For all $\psi \in N$ and $x \in \psi^{-1}(0)$, $d\psi(x) \neq 0$ and $L(\psi)(x)$ is positive definite.

2. For all $\psi \in N$, $E(s) \cap \{z\colon \psi(z) < 0\}$ is Stein, $0 < s \leq r$.

Proof. Choose $R > 0$ so that ϕ is defined on an open neighbourhood of $\overline{E(R)}$. Certainly $\phi|E(R) \in C^2_B(E(R))$ and moreover there exist $C(\phi), M(\phi) > 0$ such that for all $z \in \phi^{-1}(0) \cap E(R)$ we have

$$\|d\phi(z)\| > C(\phi) \qquad \ldots.(*)$$

$$L(\phi)(z)(v) > M(\phi)\|v\|^2, \; v \in \mathbb{C}^n.$$

From now on assume that ϕ is defined on E(R) and that (*) holds. By scalar valued Taylor's theorem, we have for $y \in \phi^{-1}(0)$ and $z \in E(R)$

$$\phi(z) = 2\mathrm{Re}\left(\sum_i \frac{\partial\phi}{\partial z_i}(y)(z_i-y_i) + \sum_{i,j} \frac{\partial^2\phi}{\partial z_i \partial z_j}(y)(z_i-y_i)(z_j-y_j)\right)$$

$$+ L(\phi)(y)(z-y) + R_\phi(z,y),$$

where $R_\phi(z,y) = 1/2\,(D^2\phi_{y+\theta(z-y)} - D^2\phi_y)((z-y)^2$ and $0 < \theta < 1$. Since ϕ is C^2, there exists $r > 0$ such that $\|D^2\phi_y - D^2\phi_0\| \leq M(\phi)/2$ for all $y \in E(r)$. Consequently for $y \in \phi^{-1}(0) \cap E(r)$, $z \in E(r)$ we have

$$|R_\phi(z,y)| \leq \frac{M(\phi)}{2}\|z-y\|^2.$$

Hence

$$(\#)\ldots\quad L(\phi)(y)(z-y) + R_\phi(z,y) > \frac{M(\phi)}{2}\|z-y\|^2,\ y \in \phi^{-1}(0) \cap E(r),\ z \in E(r).$$

Estimates (*), (#) are open conditions in $C^2_B(E(r))$ and so there exists an open neighbourhood N of ϕ in $C^2_B(E(r))$, such that (*), (#) hold for all $\psi \in N$. We claim that our choice of r, N implies the remaining statement of the Lemma. Let $\psi \in N$. We must prove that $D_s(\psi) = E(s) \cap \{z: \psi(z) < 0\}$ is Stein, $0 < s \leq r$. For this it is enough to prove that $D_s(\psi)$ is holomorphically convex. Suppose that $\{y_n: n \geq 1\}$ is a discrete subset of $D_s(\psi)$ converging to the point $y \in \partial D_s(\psi)$. If $y \in \bar{E}(s)$, there certainly exists $f \in A(E(s))$ which is unbounded on $\{y_n\}$. So suppose $\psi(y) = 0$ and let

$$F_y(z) = \sum_i \frac{\partial\psi}{\partial z_i}(y)(z_i-y_i) + \sum_{i,j} \frac{\partial^2\psi}{\partial z_j \partial z_j}(y)(z_i-y_i)(z_j-y_j).$$

The quadratic polynomial $F_y(z)$ is non-vanishing in $D_s(\psi)$. Indeed, if $F_y(z) = 0$ and $\psi(z) < 0$ we would have from the Taylor expansion of ψ at y that $L(\psi)(y)(z-y) + R_\psi(z,y) < 0$, violating (#). Observing that $F_y(y) = 0$, we see that $F_y^{-1} \in A(D_s(\psi))$ and is unbounded on any sequence of points of $D_s(\psi)$ converging to y. Hence $D_s(\psi)$ is holomorphically convex, $0 < s \leq r$. □

Theorem 7.4.2. (Grauert [3]). Let M be an s.L.p. domain in $\hat{M}$. Then for any coherent sheaf F on $\hat{M}$ we have $\dim_{\mathbb{C}} H^p(M,F) < \infty$, $p \geq 1$.

Proof. Let $\phi \in C^2_{\mathbb{R}}(M)$ define M. That is, we suppose $M = \{z \in \hat{M}: \phi(z) < 0\}$, $d\phi \neq 0$ on ∂M and $L(\phi)$ is positive definite on

∂M (see §10, Chapter 5). Since ∂M is compact, we may find a finite open cover $\{U_i: i = 1,\ldots,n\}$ of ∂M, biholomorphic maps $\gamma_i: U_i \to E(r_i)$ and open neighbourhoods N_i of $\phi\gamma_i^{-1} \in C^2_{B_1}(E(r_i))$ such that the conclusions of Lemma 7.4.1 hold for $\phi\gamma_i^{-1}$, r_i and N_i, $i = 1,\ldots,n$. In particular, the open sets U_i (respectively $U_i \cap M$) will be Stein open subsets of $\hat{M}$ (respectively M). Choose $0 < s_i < r_i$ so that $\{V_i = \gamma_i^{-1}(E(s_i)): i = 1,\ldots,n\}$ is an open cover of ∂M. By the openness assertions of Lemma 7.4.1, we may clearly inductively choose positive functions $\eta_i \in C_c^\infty(U_i)$, $1 \le i \le n$, such that

1. $\left(\phi - \sum_{i=1}^{r} \eta_i\right)\gamma_j^{-1} \in N_j$, $r,j = 1,\ldots,n.$

2. η_i is strictly positive on $V_i \cap \partial M$.

Set $\phi_0 = \phi$, $\phi_j = \phi - \sum_{i=1}^{j} \eta_i$, $1 \le j \le n$. Observe that, $d\phi_j \ne 0$ on $\phi_j^{-1}(0)$ and $L(\phi_j)$ is positive definite on $\phi_j^{-1}(0)$, $0 \le j \le n$. Set $M_0 = M$, $M_j = \{z: \phi_j(z) < 0\}$. We see that

$$M = M_0 \subset M_1 \subset \ldots \subset M_n \subset \bigcup_{i=1}^{n} U_i .$$

Moreover, M is relatively compact in M_n since ϕ_n is strictly negative on ∂M.

Step 1. The natural restriction map $H^p(M_{j+1},F) \to H^p(M_j,F)$ is surjective, $p \ge 1$.

Fix j, $0 \le j < n$. For $1 \le i \le n$, define

$$W_i = U_i \cap M_j$$

$$W_i' = U_i \cap M_j,\ i \ne j+1$$

$$= U_{j+1} \cap M_{j+1},\ i = j+1.$$

Observe that $W_i = W_i'$ unless $i = j+1$ and that, by Lemma 7.4.1, W_i, W_i' are Stein open subsets of $\hat{M}$. Adjoin Stein open subsets $W_{n+1} = W_{n+1}', \ldots, W_p = W_p'$ of M so that $\mathcal{W} = \{W_1,\ldots,W_p\}$, $\mathcal{W}' = \{W_1',\ldots,W_p'\}$ are open covers of M_j, M_{j+1} respectively.

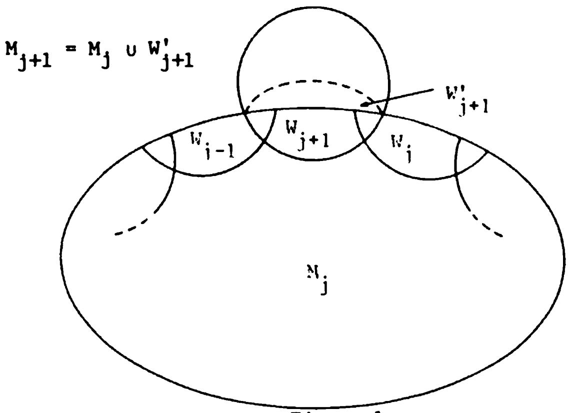


Figure 1.

The covers $\mathcal{W}$, $\mathcal{W}'$ are Leray covers of M_j, M_{j+1} for F. Observe that any p-fold intersection of distinct elements of $\mathcal{W}$ is equal to a p-fold intersection of elements of $\mathcal{W}'$ provided only that $p > 1$. Hence

$$Z^p(\mathcal{W},F) \approx Z^p(\mathcal{W}',F), \; p \geq 1.$$

Therefore the natural restriction map $H^p(\mathcal{W}',F) \to H^p(\mathcal{W},F)$ is surjective. Step 1 now follows by Leray's theorem.

Step 2. For $1 \leq j \leq n$, set $W_j = V_j \cap M$, $W'_j = U_j \cap M_n$ and observe that W_j is a relatively compact subset of W'_j. Choose Stein open subsets $W_{n+1}, W'_{n+1}, \ldots, W_p, W'_p$ of M so that W_j is a relatively compact subset of W'_j, $n+1 \leq j \leq p$, and $\mathcal{W} = \{W_1,\ldots,W_p\}$, $\mathcal{W}' = \{W'_1,\ldots,W'_p\}$ are open covers of M, M_n respectively. Just as in the proof of the finiteness theorem of Cartan-Serre the restriction map

$$R\colon Z^p(\mathcal{W}',F) \to Z^p(\mathcal{W},F)$$

is a compact operator between Fréchet spaces. From Step 1, the restriction map $H^p(\mathcal{W}',F) \to H^p(\mathcal{W},F)$ is surjective, $p \geq 1$, and so, for $p \geq 1$, the map

$$D \oplus R\colon C^{p-1}(\mathcal{W},F) \oplus Z^p(\mathcal{W}',F) \to Z^p(\mathcal{W},F)$$

is a surjective map between Fréchet spaces. Taking $A = D \oplus R$, $B = -0 \oplus R$ in Schwartz' finiteness theorem we deduce the finiteness of $H^p(\mathcal{W}, \mathcal{F})$, $p \geq 1$. □

Theorem 7.4.3. (Grauert [3]). An s.L.p. domain is holomorphically convex.

Proof. Let M be an s.L.p. domain in $\hat{M}$. It is enough to show that given $p \in \partial M$, there exists $f \in A(M)$ which is unbounded on any sequence of points of M converging to p. The method described in Exercise 2, §2 will not work here as the ideal sheaf of an infinite discrete subset of M does not extend to a coherent sheaf on $\hat{M}$. Our proof follows that in R. Narasimhan [3]. As in the proof of Theorem 7.4.2 we suppose M is defined by the C^2 function ϕ. By Lemma 7.4.1, we may choose an open neighbourhood U of p in M, biholomorphic map $\gamma: U \to E(r)$ and neighbourhood N of $\phi\gamma^{-1}$ in $C^2_B(E(r))$ such that for all $\psi\gamma^{-1} \in N$, $D_r(\psi) = \{z: \psi(z) < 0\}$ is Stein. Choose $\eta \in C^\infty_c(U)$ such that η is positive, $\eta(p) > 0$ and $(\phi - \eta)\gamma^{-1} \in N$. Let $\tilde{M} = \{z \in \hat{M}: \phi(z) - \eta(z) < 0\}$. Certainly $\tilde{M} \supset M$. As in the proof of Lemma 7.4.1, there exists $F \in A(U)$ such that $F(p) = 0$ and F is non-zero in $D_r(\phi)$. Take the open cover $\mathcal{U} = \{U \cap \tilde{M}, \tilde{M} \setminus F^{-1}(0)\}$ of $\tilde{M}$. Since the natural map $H^1(\mathcal{U}, \mathcal{O}_{\tilde{M}}) \to H^1(\tilde{M}, \mathcal{O}_{\tilde{M}})$ is injective (Exercise 4, §3, Chapter 6), and $\tilde{M}$ is s.L.p., we have $\dim_{\mathbb{C}} H^1(\mathcal{U}, \mathcal{O}_{\tilde{M}}) < \infty$. Let L denote the infinite dimensional linear subspace of $Z^1(\mathcal{U}, \mathcal{O}_{\tilde{M}})$ defined by

$$L = \left\{\sum_{j=1}^{\infty} c_j F^{-j} : c_j \in \mathbb{C} \text{ and all but finitely many } c_j\text{'s are zero}\right\}.$$

Since $\dim_{\mathbb{C}} H^1(\mathcal{U}, \mathcal{O}_{\tilde{M}}) < \infty$, there exist elements of L which are boundaries. Therefore we may find a combination

$$G = \sum_{j=1}^{n} g_j F^{-j} \in L,$$

with not all the g_j's vanishing, such that $G = D(H)$, for some $H \in C^0(\mathcal{U}, \mathcal{O}_{\tilde{M}})$. Hence there exist $H_0 \in \mathcal{O}(U \cap \tilde{M})$, $H_1 \in \mathcal{O}(\tilde{M} \setminus F^{-1}(0))$ with $G = H_1 - H_0$ on $(U \cap \tilde{M}) \setminus F^{-1}(0)$. That is, we have

$$H_1 = H_0 + \sum_{j=1}^{n} g_j F^{-1} \text{ on } (U \cap \tilde{M}) \setminus F^{-1}(0).$$

Clearly $f = H_1|M \in A(M)$ is unbounded on every sequence of points of M converging to p. □

We may now give a solution to Levi's problem.

Theorem 7.4.4. An s.L.p. domain of a Stein manifold is Stein.

Proof. Immediate from Theorem 7.4.3. □

Remarks.

1. Suppose that Ω is a Levi pseudoconvex domain in $\mathbb{C}^n$ which is not s.L.p. Then it can easily be proved that Ω is the union of an increasing family of s.L.p. domains (see Oka [1] or Gunning and Rossi [1; Lemma 2, Section D, Chapter 10]). Hence, by Theorem 7.4.4., Ω is the limit of an increasing family of domains of holomorphy and so, by a theorem of Behnke and Stein [1], a domain of holomorphy (see also Gunning and Rossi [1; section D, Chapter 10]). This result was first obtained by Oka in case n = 2 and then for general n independently by Oka [1], Bremermann [1] and Norguet [1]. Their result generalises to the case when Ω is a Levi pseudoconvex domain of a Riemann domain spread over $\mathbb{C}^n$ (see Gunning and Rossi [1; section D, Chapter 10]). Moreover, it is not necessary to assume that the boundary of Ω is smooth. All that is required is that the function $-\log(d(z,\partial\Omega))$ is plurisubharmonic in Ω (see the above references and also Hörmander [1]). It is not generally true that a Levi pseudoconvex domain of an arbitrary complex manifold is holomorphically convex. For an example of a non-holomorphically convex Levi pseudoconvex domain see Grauert [4] and also the survey article by Siu [1], especially §7.

2. R. Narasimhan [4] has shown that if M is a complex manifold then $\dim H^p(M,\mathcal{F}) < \infty$, $p \geq 1$, for all coherent sheaves $\mathcal{F}$ on M if and only if M is holomorphically convex. The proof that finiteness of cohomology implies holomorphic convexivity is similar to that indicated in Exercise 2, §2 and makes use of the observation that the space of bounded infinite sequences of complex numbers is of infinite codimension in the space of all infinite complex sequences. The converse is much deeper and uses Grauert's direct image theorem together with results of H. Cartan [3] and Remmert [1] to the effect that if M is holomorphically convex then M possesses a maximal non-

trivial compact analytic subset, A. Narasimhan proves that the inclusion $A \to M$ induces isomorphisms $H^p(A,F_A) \to H^p(M,F)$, $p \geq 1$. The finiteness theorem of Cartan-Serre for compact analytic spaces then gives the result. In particular, M will be Stein if and only if M has no non-trivial compact analytic subsets. See Rossi [2] for a discussion of the case of s.L.p. domains.

3. Suppose M is a strictly pseudoconvex manifold. That is, there exists a C^∞ function $\phi: M \to \mathbb{R}$ such that $L(\phi)$ is everywhere positive definite and $M_a = \{z \in M: \phi(z) < a\}$ is relatively compact for all $a \in \mathbb{R}$. By Sard's theorem, M_a is s.L.p. for a dense set $\Sigma \subset \mathbb{R}$ of values of a. Now it can be shown (see Gunning and Rossi [1; Section C, Chapter 10]) that plurisubharmonic functions satisfy a maximal principle and so M cannot have any non-trivial compact analytic subsets. It follows from Remark 2 that M_a is Stein, $a \in \Sigma$. Thus we have expressed M as a union of an increasing family of Stein manifolds parametrized by points in a dense subset of $\mathbb{R}$. It is shown in Docquier and Grauert [1] that this is enough to prove M Stein. See also Siu [1] and note that it is not generally true that an increasing union of Stein open sets, parametrized by the positive integers, need be Stein. Of course the union will be Stein if all the sets are domains in $\mathbb{C}^n$ or a Riemann domain (see Remark 1). If they are all subdomains of a Stein manifold it is not yet known whether their union must be Stein (see also Markoe [1]). We shall prove in Chapter 11 that every strictly pseudoconvex manifold is Stein. Our proof will depend on the existence theory for the $\bar{\partial}$-operator and follows the approach of Kohn [1], Andreotti-Vesentini [1], Hörmander [1] and Vesentini [1].

Exercise. Let M be an s.L.p. domain in the complex manifold M and suppose that M has C^2 defining function ϕ. Show

a) There exists $C > 0$ such that $M_c = \{x \in M: \phi(x) < -c\}$ is s.L.p. for $0 \leq c < C$.

b)* If M has no non-trivial compact analytic subvarieties then M_c is Stein, $0 < c < C$ (See Rossi [1] and note that we do not need to assume a maximal principle for strictly psh functions).

§5. Coherent sheaves on projective space.

In this section we shall prove theorems A and B of Serre for coherent sheaves on projective space. These fundamental theorems play a similar rôle in the theory of projective varieties to that of Cartan's theorems A and B in Stein manifold theory.

Throughout this section $\mathcal{U} = \{U_i\}$ will denote the standard open cover of $P^n(\mathbb{C})$ by the open sets $U_i = \{(z_0,\ldots,z_n): z_i \neq 0\}$, $0 \le i \le n$. Suppose that F is a coherent sheaf on $P^n(\mathbb{C})$. Since each U_i is biholomorphic to $\mathbb{C}^n$ and is therefore Stein, $\mathcal{U}$ is a Leray cover of $P^n(\mathbb{C})$ for F. It follows immediately from Leray's theorem that

$$H^p(P^n(\mathbb{C}),F) = 0, \ p > n.$$

Let H denote the hyperplane section bundle of $P^n(\mathbb{C})$. Relative to the cover $\mathcal{U}$, H has transition functions $\phi_{ij} = z_j/z_i$. For $m \in \mathbb{Z}$, we let H^m denote the holomorphic line bundle with transition functions $(z_j/z_i)^m$ on U_{ij}. We may regard H^m as the line bundle associated to the divisor $z_0^m = 0$. That is, if we let $P^{n-1}(\mathbb{C}) \subset P^n(\mathbb{C})$ denote the hyperplane $z_0 = 0$, we have $H^m = [m.P^{n-1}(\mathbb{C})]$. With this convention, H^m has the "canonical" section s^m given locally by $s_i^m = (z_0/z_i)^m$ and $\mathrm{div}(s^m) = m.P^{n-1}(\mathbb{C})$.

As is conventional, we let $\mathcal{O}(m)$ denote the sheaf of germs of holomorphic sections of H^m, $m \in \mathbb{Z}$.

Lemma 7.5.1. Let Σ be a hyperplane in $P^n(\mathbb{C})$. For $m \in \mathbb{Z}$, we have a non-zero $\mathcal{O}$-morphism

$$\phi_\Sigma: \mathcal{O}(m) \to \mathcal{O}(m+1)$$

satisfying $\phi_{\Sigma,z} = 0$ if and only if $z \in \Sigma$. In particular, ϕ_Σ restricts to an isomorphism of $\mathcal{O}(m)$ with $\mathcal{O}(m+1)$ over $P^n(\mathbb{C}) \setminus \Sigma$.

Proof. Let Σ have equation $s(z_0,\ldots,z_n) = 0$. Define $\phi_\Sigma(f) = s_z f$, $f \in \mathcal{O}(m)_z$. □

Remark. The morphism ϕ_Σ is given locally by $\phi_\Sigma(f_i) = f_i s/z_i$, $f_i \in \mathcal{O}(m)(U_i)$.

Let $P^{(m)}(\mathbb{C}^{n+1})$ denote the space of homogeneous polynomials of degree m on $\mathbb{C}^{n+1}$. Then (Proposition 5.9.2),

$$H^0(P^n(\mathbb{C}),\mathcal{O}(m)) \approx P^{(m)}(\mathbb{C}^{n+1}),\ m \geq 0$$

$$= 0,\ m < 0.$$

Theorem 7.5.2.

1. For $m \geq 0$, we have

$$H^p(P^n(\mathbb{C}),\mathcal{O}(m)) = 0,\ p \neq 0$$

$$\approx P^{(m)}(\mathbb{C}^{n+1}),\ p = 0.$$

2. For $m < 0$, we have

$$H^p(P^n(\mathbb{C}),\mathcal{O}(m)) = 0,\ p \neq n$$

$$\approx P^{(-m-n-1)}(\mathbb{C}^{n+1}),\ p = n.$$

Proof. Since we have already covered the cases $p = 0$, $p > n$, we shall assume from now on that $1 \leq p \leq n$. Let us start by considering the case $m = 0$. Suppose $c \in Z^p(\mathcal{U},\mathcal{O})$. Then $c = \{c(s) = c(s_0,\ldots,s_p)\}$, where $c(s)\colon U_s \to \mathbb{C}$ is holomorphic and $s = (s_0,\ldots,s_p)$ is a $(p+1)$-tuple of (distinct) integers lying between 0 and n. We may regard each $c(s)$ as a holomorphic function on $U_s \subset \mathbb{C}^{n+1}$ which is homogeneous of degree zero. By Theorem 2.1.10, we may take Laurent expansions of the $c(s)$. Thus

$$c(s)(z_0,\ldots,z_n) = \sum_{r_0+\ldots+r_n=0} c(s)_{r_0\ldots r_n} z_0^{r_0}\ldots z_n^{r_n}$$

$$= \sum_{|r|=0} c(s)_r z^r, \text{ using multi-index notation.}$$

Observe that the coefficient $c(s)_{r_0\ldots r_n}$ will be zero if any $r_j < 0$ with $j \notin \{s_0,\ldots,s_p\}$. Define

$$a(s_0,\ldots,s_{p-1}) = \frac{1}{n-p+1}\sum{}' c(j,s_0,\ldots,s_{p-1})^+,$$

where Σ' denotes the sum over all $j \notin \{s_0,\dots,s_{p-1}\}$ and $c(j,s_0,\dots,s_{p-1})^+$ is the holomorphic function on $U_{s_0} \cap \dots \cap U_{s_{p-1}}$ with Laurent series

$$\sum_{\substack{|r|=0 \\ r_j \geq 0}} c(j,s_0,\dots,s_{p-1})_r z^r .$$

That is, we omit terms from the Laurent expansion of $c(j,s_0,\dots,s_{p-1})$ having $r_j < 0$. Our construction defines an element $a \in C^{p-1}(\mathcal{U},\mathcal{O})$. Now

$$(Da)_{s_0\dots s_p} = \frac{1}{n-p+1}\left(\sum{}'(c(j,s_1,s_2,\dots,s_p) - \dots \pm c(j,s_0,\dots,s_{p-1}))^+\right)$$

$$= \frac{1}{n-p+1}\left(\sum{}' c(s_0,\dots,s_p)\right)^+, \text{ since } Dc = 0$$

$$= c(s_0,\dots,s_p).$$

We have shown that $Z^p(\mathcal{U},\mathcal{O}) = DC^{p-1}(\mathcal{U},\mathcal{O})$ and so $H^p(\mathcal{U},\mathcal{O}) = 0$, $p > 0$.

Exactly the same proof shows that if $m > 0$ then $H^p(\mathcal{U},\mathcal{O}(m)) = 0$, $p > 0$. Indeed, the only difference is that a cochain will now be a collection of holomorphic functions which are homogeneous of degree m. The same proof also works for $m < 0$ provided that $p < n$. We conclude by considering the case $p = n$, $m < 0$. Suppose $c \in Z^n(\mathcal{U},\mathcal{O}(m))$. Then

$$c\colon U_{01\dots n} \to \mathbb{C}$$

is holomorphic and homogeneous of degree m. Thus

$$c(z_0,\dots,z_n) = \sum_{|r|=m} c_r z^r .$$

Write $c = C_0 + C_1$, where C_0 is the sum over all terms $c_{r_0\dots r_n} z_0^{r_0}\dots z_n^{r_n}$ where at least one index r_j is positive and C_1 is the sum over terms such that every index r_j is negative. Notice that $C_1 \equiv 0$ if $-m \leq n$. As above it is easily seen that $C_0 = Da$ for some $(n-1)$-cochain a. A simple Laurent series argument shows that a non-zero C_1 can never be a coboundary. So suppose $m \leq -n-1$ and set

$$E = \left\{ \sum_{|r|=m} c_r z^r : c_r \in \mathbb{C}, \text{ every index } r_j < 0 \right\}.$$

By what we have shown above $E \approx H^n(\mathcal{U},\mathcal{O}(m))$. Given $c \in E$, we may write

$$c = (z_0 \ldots z_n)^{-1} \sum_{\substack{r_0,\ldots,r_n \le 0 \\ r_0+\ldots+r_n = m+m+1}} c_r z_0^{r_0+1} \ldots z_n^{r_n+1} .$$

It follows that $H^n(\mathcal{U},\mathcal{O}(m)) \approx P^{(-m-n-1)}(\mathbb{C}^{n+1})$ where the isomorphicm is given explicitly by mapping $P \in P^{(-m-n-1)}(\mathbb{C})$ to

$$c(z_0,\ldots,z_n) = (z_0 \ldots z_n)^{-1} P(z_0^{-1},\ldots,z_n^{-1}) . \qquad \square$$

Remark. For an alternative proof of Theorem 7.5.2, using non-trivial facts from the Hodge theory of Kähler manifolds, see Seminar 18 by Serre in H. Cartan [2].

Given a coherent sheaf F on $P^n(\mathbb{C})$ we let

$$F(m) = F \otimes_{\mathcal{O}} \mathcal{O}(m), \; m \in \mathbb{Z} .$$

We call $F(m)$ the sheaf F "twisted by $\mathcal{O}(m)$". One feature of twisting is that we expect $\dim_{\mathbb{C}} H^0(P^n(\mathbb{C}),F(m))$ to be an increasing function of m. To explain why this should be so, let us consider the case when F is the sheaf of sections of a holomorphic vector bundle E. We claim that $H^0(P^n(\mathbb{C}),E(m)) \cong \{s \in M(E) : \mathrm{div}(s) + m.P^{n-1}(\mathbb{C}) \ge 0\}$. Certainly it follows from this isomorphism that $\dim_{\mathbb{C}} H^0(P^n(\mathbb{C}),E(m))$ is an increasing function of m. Suppose that E has transition functions ϕ_{ab} relative to a cover $\mathcal{W}$ of $P^n(\mathbb{C})$ where we suppose that $\mathcal{W}$ is a refinement of $\mathcal{U}$. Let $s \in M^*(E)$ and $\mathrm{div}(s) + m.P^{n-1}(\mathbb{C}) \ge 0$. If $s_a \in M^*(W_a)$ is the local representative of s on $W_a \subset U_{i(a)}$, we have

$$(z_0/z_{i(a)})^m s_a \in A(W_a).$$

Clearly if $W_b \subset U_{i(b)}$, we have

$$\phi_{ab}(z_{i(b)}/z_{i(a)})^m (z_0/z_{i(b)})^m s_b = (z_0/z_{i(a)})^m s_a \text{ on } W_{ab}.$$

Therefore $\{(z_0/z_{i(a)})^m s_a\}$ defines a holomorphic section of E(m). Reversing the argument shows that every holomorphic section of E(m) gives rise to a meromorphic section of E such that $div(s) + m.P^{n-1}(\mathbb{C}) \geq 0$. Of course, our argument depended on being able to show that E admits at least one non-trivial meromorphic section.

Remark. We should point out that a holomorphic vector bundle E on $P^n(\mathbb{C})$ always restricts to a holomorphically trivial bundle over U_i, $0 \leq i \leq n$, Serre [2]. This fact also follows from a general and difficult theorem of Grauert to the effect that a holomorphic vector bundle on a Stein manifold is holomorphically trivial if and only if it is topologically trivial. See also Adams and Griffiths [1] for a proof that holomorphic vector bundles over polydiscs in $\mathbb{C}^n$ are holomorphically trivial as well as references and discussion concerning Grauert's theorems.

Theorem 7.5.3. (Theorems A and B of Serre). Let F be a coherent sheaf on $P^n(\mathbb{C})$. Then there exists $m_0 = m_0(F) \in \mathbb{Z}$ such that

A. For each $z \in P^n(\mathbb{C})$, $H^0(P^n(\mathbb{C}),F(m))$ generates $F(m)_z$ as a $\mathcal{O}_z$-module, $m \geq m_0$.

B. For $m \geq m_0$, $H^p(P^n(\mathbb{C}),F(m)) = 0$, $p \geq 1$.

Proof. (Seminar 19 by Serre, H. Cartan [2]). We start by looking at some special cases. Suppose $F \cong \mathcal{O}(q)$. Take $m_0 = -q$. Since $\mathcal{O}(q)(m) = \mathcal{O}(q+m)$, we see immediately from Theorem 7.5.2 that $H^p(P^n(\mathbb{C}),F(m)) = 0$, $p \geq 1$, $m \geq m_0$. Clearly for all $z \in P^n(\mathbb{C})$, $H^0(P^n(\mathbb{C}),F(m))$ generates $F(m)_z$, $m \geq m_0$. Since cohomology commutes with direct sums, A and B hold whenever $F \cong \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$ and we may take $m_0 = -\min\{a_i\}$.

We prove the theorem in general by an induction on n. Let A_n, B_n denote statements A and B for dimension n. We shall show that A_{n-1} and B_{n-1} imply A_n and A_n implies B_n. The Theorem is, of course, trivial for $n = 0$.

Step 1. A_{n-1} and B_{n-1} imply A_n. Let $z \in P^n(\mathbb{C})$ and choose any hyperplane $\Sigma \subset P^n(\mathbb{C})$ not containing z. For $m \in \mathbb{Z}$, let

$\tilde{\phi}_\Sigma = \mathrm{Id} \otimes \phi_\Sigma$: $F(m) \to F(m+1)$, where ϕ_Σ is the map given by Lemma 7.5.1. Observe that $\tilde{\phi}_\Sigma$ restricts to an isomorphism of $F(m)$ with $F(m+1)$ over $P^n(\mathbb{C}) \setminus \Sigma$. Suppose $H^0(P^n(\mathbb{C}), F(m_0))$ generates $F(m_0)_z$. Then $H^0(P^n(\mathbb{C}), F(m))$ generates $F(m)_z$ for $m \geq m_0$. Indeed, $\tilde{\phi}_\Sigma^{m-m_0}$: $F(m_0) \to F(m)$ restricts to an isomorphism over $P^n(\mathbb{C}) \setminus \Sigma$ and so maps any set of generators for $F(m_0)_z$ to a set of generators for $F(m)_z$. Let $A_n(z)$ (respectively $A_n'(z)$) be the statement "There exists $m_0 = m_0(F, z)$ such that $H^0(P^n(\mathbb{C}), F(m_0))$ (respectively $H^0(P^n(\mathbb{C}), F(m))$) generates $F(m_0)_z$ (respectively $F(m)_z$, $m \geq m_0$)". By the coherence of F, $A_n(z)$ implies $A_n(y)$ for y in some open neighbourhood of z. Since $A_n(y)$ implies $A_n'(y)$, it follows from the compactness of $P^n(\mathbb{C})$ that $A_n(z)$ for all $z \in P^n(\mathbb{C})$ impies A_n. Therefore it is enough to prove that A_{n-1} and B_{n-1} imply $A_n(z)$.

Without loss of generality suppose $z \in P^{n-1}(\mathbb{C}) \subset P^n(\mathbb{C})$. Set $\phi = \tilde{\phi}_{P^{n-1}(\mathbb{C})}$: $F(-1) \to F$ and let

$$K = \mathrm{Ker}\,\phi: F(-1) \to F$$

$$G = \mathrm{Coker}\,\phi: F(-1) \to F.$$

We have the exact sequence

$$0 \to K \to F(-1) \xrightarrow{\phi} F \to G \to 0.$$

Tensoring with $\mathcal{O}(m)$ we obtain the exact sequence

$$0 \to K(m) \to K(m-1) \xrightarrow{\phi_m} F(m) \to G(m) \to 0.$$

Since $\phi_m | U_0$ is an isomorphism we see that $K_{U_0} = G_{U_0} = 0$. Let i denote the inclusion map of $P^{n-1}(\mathbb{C})$ in $P^n(\mathbb{C})$. For any sheaf H on $P^n(\mathbb{C})$ we let $H^* = i^{-1}H$ denote the restriction of H to $P^{n-1}(\mathbb{C})$. Observe that $I_{P^{n-1}(\mathbb{C})} \subset \mathcal{O}_{P^n(\mathbb{C})}$ acts trivially on K and G. Indeed if $s_z \in I_{P^{n-1}(\mathbb{C})}$, $f_z \in K$, then $s_z f_z = 0$ since $K = \mathrm{Ker}(\phi)$. Similarly for G since $G = \mathrm{Coker}(\phi)$. Hence K^*, G^* have the natural structure of $\mathcal{O}_{P^{n-1}(\mathbb{C})} = \mathcal{O}^*_{P^n(\mathbb{C})} / I^*_{P^{n-1}(\mathbb{C})}$ -modules and the reader may easily verify that K^*, G^* are coherent $\mathcal{O}_{P^{n-1}(\mathbb{C})}$-modules. Moreover, it is clear that $K(m)^* \approx K^*(m)$, $G(m)^* \approx G^*(m)$, $m \in \mathbb{Z}$.

By Exercise 12, §3, Chapter 6, $H^p(P^n(\mathbb{C}),K(m)) = H^p(P^{n-1}(\mathbb{C}),K(m)^*)$, $p \geq 0$, similarly for $G(m)$. Applying our inductive assumption to K^*, G^*, there exists $m_1 \in \mathbb{Z}$ such that for $p \geq 1$

$$H^p(P^n(\mathbb{C}),K(m)) = H^p(P^{n-1}(\mathbb{C}),K^*(m)) = 0, \ m \geq m_1$$

$$H^p(P^n(\mathbb{C}),G(m)) = H^p(P^{n-1}(\mathbb{C}),G^*(m)) = 0, \ m \geq m_1.$$

We have the short exact sequences

$$0 \to K(m) \to F(m-1) \to \mathrm{Im}(\phi_{m-1}) \to 0$$

$$0 \to \mathrm{Im}(\phi_{m-1}) \to F(m) \to G(m) \to 0.$$

Taking the cohomology sequences of these short exact sequences we obtain the exact sequences

$$H^1(P^n(\mathbb{C}),F(m-1)) \to H^1(P^n(\mathbb{C}),\mathrm{Im}(\phi_{m-1})) \to H^2(P^n(\mathbb{C}),K(m))$$

$$H^1(P^n(\mathbb{C}),\mathrm{Im}(\phi_{m-1})) \to H^1(P^n(\mathbb{C}),F(m)) \to H^1(P^n(\mathbb{C}),G(m)).$$

Since $H^1(P^n(\mathbb{C}),G(m))$, $H^2(P^n(\mathbb{C}),K(m)) = 0$, $m \geq m_1$, we see that $\dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),F(m-1)) \geq \dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),\mathrm{Im}(\phi_{m-1})) \geq \dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),F(m))$, for $m \geq m_1$. Hence $\dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),F(m))$ is a decreasing function of m, $m \geq m_1$. But now by the finiteness theorem of Cartan-Serre, $\dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),F(m)) < \infty$, and so we deduce that $\dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),F(m))$ is independent of m for $m > m_2$, say. In particular, for $m \geq m_2$ we have

$$\dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),F(m-1)) = \dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),\mathrm{Im}(\phi_{m-1})) = \dim_{\mathbb{C}}H^1(P^n(\mathbb{C}),F(m)).$$

Since the surjective homomorphism $H^1(P^n(\mathbb{C}),\mathrm{Im}(\phi_{m-1})) \to H^1(P^n(\mathbb{C}),F(m))$ is between spaces of the same finite dimension, it must be injective, $m \geq m_2$. Therefore, from the cohomology sequence we see that $H^0(P^n(\mathbb{C}),F(m)) \to H^0(P^n(\mathbb{C}),G(m))$ is surjective, $m \geq m_2$. But $H^0(P^n(\mathbb{C}),G(m)) = H^0(P^{n-1}(\mathbb{C}),G^*(m))$ and so by A_{n-1} we see that $H^0(P^n(\mathbb{C}),G(m))$ generates $G^*(m)_z = G(m)_z$ for $m \geq m_0$, where we may suppose $m_0 \geq m_2$. In particular, $H^0(P^n(\mathbb{C}),F(m))$ generates $G(m)_z$ for

$m \geq m_0$. We claim that for $m \geq m_0$, $H^0(P^n(\mathbb{C}),F(m))$ generates $F(m)_z$. Suppose $z \in U_i$, $i \neq 0$. Let $\psi: F \to F(1)$ be the homomorphism defined by $\psi(f) = z_i f$. As described above, ψ^{-m} restricts to an isomorphism of $F(m)_{U_i}$ with F_{U_i}. Under this identification of $F(m)_{U_i}$ with F_{U_i}, the map ϕ corresponds to multiplication by $t_0 = z_0/z_i$ and $G(m)_z$ is therefore identified with F_z/t_0F_z. Let $M_z \subset F_z$ be the submodule generated by the elements of $H^0(P^n(\mathbb{C}),F(m))$. Since $G(m)_z$ is generated by the elements of $H^0(P^n(\mathbb{C}),F(m))$, $m \geq m_0$, the image of M_z in F_z/t_0F_z is the whole of F_z/t_0F_z. That is, $F_z = t_0F_z + M_z$. Since $t_0 \in m_z \subset O_z$, it follows from Nakayama's lemma that $F_z = M_z$.

Step 2. A_n implies B_n.

For $p > n$, we have already shown that $H^p(P^n(\mathbb{C}),F(m)) = 0$ for all m. We prove B_n by decreasing induction on p. So suppose that for every coherent sheaf F on $P^n(\mathbb{C})$, there exists $m_0 = m_0(F, p)$ such that $H^{p+1}(P^n(\mathbb{C}),F(m)) = 0$, $m \geq m_0$. By A_n there exists $m \in \mathbb{Z}$ such that $H^0(P^n(\mathbb{C}),F(m))$ generates $F(m)_z$ for all $z \in P^n(\mathbb{C})$. Choose a basis $s_1,\ldots,s_k$ of $H^0(P^n(\mathbb{C}),F(m))$. We have the exact sequence

$$0 \to K \to O^k \xrightarrow{s} F(m) \to 0,$$

where $s = (s_1,\ldots,s_k)$ and $K = \mathrm{Ker}(s)$. Tensoring by $O(q)$ yields the exact sequence

$$0 \to K(q) \to O^k(q) \to F(m+q) \to 0$$

and corresponding portion of the cohomology sequence

$$H^p(P^n(\mathbb{C}),O^k(q)) \to H^p(P^n(\mathbb{C}),F(m+q)) \to H^{p+1}(P^n(\mathbb{C}),K(q)).$$

By the special case of B described at the beginning of the proof, $H^p(P^n(\mathbb{C}),O^k(q)) = 0$ for $p > 0$, $q \geq 0$. By our inductive assumption, $H^{p+1}(P^n(\mathbb{C}),K(q)) = 0$ for sufficiently large q. Hence $H^p(P^n(\mathbb{C}),F(m+q)) = 0$ for sufficiently large q. This completes the inductive step and the proof of the theorem. □

Remark. The reader should note the crucial rôle played by the finiteness theorem of Cartan-Serre in the proof of Theorem 7.5.3. In §6, we give another proof of Theorem 7.5.3 depending on Grauert's finiteness theorem.

Now for some applications of Serre's theorem.

Theorem 7.5.4. Every holomorphic vector bundle E on $P^n(\mathbb{C})$ has a non-trivial meromorphic section.

Proof. By statement A of Theorem 7.5.3, there exists m_0 such that $H^0(P^n(\mathbb{C}),\underline{E}(m_0))$ generates $\underline{E}(m_0)_z$ for all $z \in P^n(\mathbb{C})$. In particular, $\dim_{\mathbb{C}} H^0(P^n(\mathbb{C}),\underline{E}(m_0)) \geq 1$. But as we showed earlier, $H^0(P^n(\mathbb{C}),\underline{E}(m_0)) \cong \{s \in M(E)\colon \mathrm{div}(s) + m_0.P^{n-1}(\mathbb{C}) \geq 0\}$. □

Remark. Of course, we can prove much more than that E has a non-trivial meromorphic section. Using statement B, we can show that for any $v \in E_z$, there exists $s \in M^*(E)$ such that $s(z) = v$. See the exercises at the end of the section.

Theorem 7.5.5. (Chow's theorem). Every analytic subset of $P^n(\mathbb{C})$ is algebraic.

Proof. Let X be an analytic subset of $P^n(\mathbb{C})$ with ideal sheaf I. Then I is coherent subsheaf of O (see §1 for discussion and references). We prove: Given $z \in P^n(\mathbb{C}) \setminus X$, there exists a homogeneous polynomial $p = p(z_0,\ldots,z_n)$ which vanishes on X and does not vanish at x. Granted this, we consider the set of all homogeneous polynomials which vanish on X. The common zero locus of these polynomials is X and by the Hilbert basis theorem we may choose a finite subset of these polynomials with common zero locus X.

Suppose then that I_z denotes the ideal sheaf of $X \cup \{z\}$. For $m \in \mathbb{Z}$, we have the exact sequence

$$0 \to I_z(m) \to I \to \mathbb{C}(z)(m) \to 0 ,$$

where $\mathbb{C}(z)$ is the skyscrapper sheaf supported at z. Note that $\mathbb{C}(z)(m) = \mathbb{C}(z)$. By B of Theorem 7.5.3, there exists m_0 such that $H^1(P^n(\mathbb{C}),I_z(m) = 0$, $m \geq m_0$. Taking the cohomology sequence of our short exact sequence we obtain for $m \geq m_0$ the exact sequence

$$H^0(P^n(\mathbb{C}),\mathcal{I}(m)) \to H^0(P^n(\mathbb{C}),\mathbb{C}(z)(m)) \to 0.$$

Now $H^0(P^n(\mathbb{C}),\mathbb{C}(z)(m)) \cong \mathbb{C}$ and so there exists $p \in H^0(P^n(\mathbb{C}),\mathcal{I}(m))$ such that $p(z) \neq 0$. But since $\mathcal{I} \subset \mathcal{O}$, it follows that $H^0(P^n(\mathbb{C}),\mathcal{I}(m))$ is a subspace of $H^0(P^n(\mathbb{C}),\mathcal{O}(m)) \approx P^{(m)}(\mathbb{C}^{n+1})$. Indeed, it is just the (non-empty!) subspace of homogeneous polynomials of degree m which vanish on X. □

Remark. Chow's theorem is a special instance of a general relationship between global analytic and algebraic structures on projective space. This relationship is explained fully in Serre's G.A.G.A. paper, Serre [3]. In particular, Serre shows that there is an equivalence between coherent algebraic sheaves and coherent (analyitc) sheaves on projective space. Chow's theorem is one corollary of this correspondence. Another is that every holomorphic vector bundle on projective space is "algebraic".

Theorem 7.5.6. Let $\mathcal{F}$ be a coherent sheaf on $P^n(\mathbb{C})$. Then $\mathcal{F}$ has a resolution

$$0 \to \underline{E}_n \to \underline{E}_{n-1} \to \cdots \to \underline{E}_0 \to \mathcal{F} \to 0$$

by locally free sheaves. We may require that for $0 \leq j < n$ the sheaf $\underline{E}_j$ is isomorphic to a direct sum $\mathcal{O}(a_j)^{k_j}$.

Proof. Just as in the proof of Step 2 of Theorem 7.5.3, statement A of Theorem 7.5.3 implies that there exists an exact sequence

$$0 \to \mathcal{K}_1 \to \mathcal{O}^{k_0} \to \mathcal{F}(m_0) \to 0.$$

Setting $\mathcal{F} = \mathcal{K}_0$ and iterating this construction we obtain for $0 \leq j \leq n$ exact seqeunces

$$0 \to \mathcal{K}_{j+1} \to \mathcal{O}^{k_j} \to \mathcal{K}_j(m_j) \to 0.$$

Tensoring each sequence by an appropriate power of $\mathcal{O}(1)$ we obtain exact sequences

$$0 \to \mathcal{K}_{j+1}(-m_0-\ldots-m_j) \to \mathcal{O}^{k_j}(-m_0-\ldots-m_j) \to \mathcal{K}_j(-m_0-\ldots-m_{j-1}) \to 0$$

and hence a long exact sequence

$$0 \to \mathcal{K}_n(p_{n-1}) \to \mathcal{O}^{k_{n-1}}(p_{n-1}) \to \ldots \to \mathcal{O}^{k_0}(p_0) \to \mathcal{F} \to 0.$$

The coherence of $\mathcal{K}_n(p_{n-1})$ together with the Hilbert Syzygy theorem imply that $\mathcal{K}_n(p_{n-1})$ is locally free (see the proof of Corollary 7.1.8). □

Remarks.

1. In the exercises at the end of the section we show how, in certain casea, we can construct explicitly a resolution of a coherent sheaf on $P^n(\mathbb{C})$ by locally free sheaves.

2. To appreciate the significance of Theorem 7.5.6 we first need to discuss *Serre's duality theorem*. In Chapter 10 we shall prove Serre's duality theorem for complex manifolds: Let M be a compact complex manifold of dimension m and E be a holomorphic vector bundle on M. Then $H^p(M,\Omega^q(\underline{E}))$ is isomorphic (not canonically) to $H^{m-p}(M,\Omega^{m-q}(\underline{E}^*))$, $p,q \geq 0$. In particular, if $q = 0$, $H^p(M,\underline{E}) \cong H^{m-p}(M,\underline{K} \otimes \underline{E}^*)$, where K denotes the canonical bundle of M. Serre duality plays an important role in complex analysis. In particular, it gives an easy proof of the Riemann-Roch theorem for compact Riemann surfaces (see Chapter 10). Suppose that $M = P^n(\mathbb{C})$. Then the canonical bundle of $P^n(\mathbb{C})$ is canonically isomorphic to H^{-n-1} (§9, Chapter 5). Theorem 7.5.2 implies immediately that $H^p(P^n(\mathbb{C}),\mathcal{O}(m)) \approx H^{n-p}(P^n(\mathbb{C}),\mathcal{O}(-m) \otimes \underline{K})$, $p \geq 0$, $m \in \mathbb{Z}$. A simple computation shows that we have a (canonical) Serre duality for any sheaf on $P^n(\mathbb{C})$ which is a direct sum of sheaves $\mathcal{O}(m)$. Theorem 7.5.6, together with some basic facts from homological algebra, now allows us to verify Serre duality for any locally free sheaf on $P^n(\mathbb{C})$. Even more, we may prove a duality theorem for arbitrary coherent sheaves on $P^n(\mathbb{C})$. However, the formulation of this duality theorem will now involve "Ext" groups. These matters are explained further in Griffiths and Harris [1] and Hartshorne [1,2]. We should also mention that Ramis and Ruget [1] and Ramis, Ruget and Verdier [1] have

developed a general duality theorem applicable to analytic spaces and proper analytic maps.

For the remainder of this section we make a preliminary study of holomorphic vector bundles on projective space.

Proposition 7.5.7. The group of holomorphic line bundles on $P^n(\mathbb{C})$ is isomorphic to the infinite cyclic group generated by the hyperplane section bundle H.

Proof. First note that the group generated by H in $HLB(P^n(\mathbb{C}))$ is infinite cyclic. Indeed, H^p is isomorphic to $\underline{\mathbb{C}}$ if and only if $H^0(P^n(\mathbb{C}),\mathcal{O}(p)) \cong \mathbb{C}$ and by Theorem 7.5.2 this happens if and only if $p = 0$. It follows from the cohomology sequence of $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ and the vanishing of $H^1(P^n(\mathbb{C}),\mathcal{O})$, $H^2(P^n(\mathbb{C}),\mathcal{O})$ that

$$H^1(P^n(\mathbb{C}),\mathcal{O}^*) \cong H^2(P^n(\mathbb{C}),\mathbb{Z}),$$

where the isomorphism is given by c_1, 1st. Chern class map. By topology, $H^2(P^n(\mathbb{C}),\mathbb{Z}) \cong \mathbb{Z}$, $n \geq 1$. Let $P^1(\mathbb{C}) \subset P^n(\mathbb{C})$. We have the commutative diagram

$$\begin{array}{ccc} H^1(P^n(\mathbb{C}),\mathcal{O}^*) & \xrightarrow{c_1} & H^2(P^n(\mathbb{C}),\mathbb{Z}) \cong \mathbb{Z} \\ \downarrow r & & \downarrow \\ H^1(P^1(\mathbb{C}),\mathcal{O}^*) & \xrightarrow{c_1} & H^2(P^1(\mathbb{C}),\mathbb{Z}) \cong \mathbb{Z} \end{array}$$

where the vertical maps are induced by inclusion. Now we know from Example 13, §3, Chapter 6 that H generates $H^1(P^1(\mathbb{C}),\mathcal{O}^*)$. But r maps the hyperplane section bundle of $P^n(\mathbb{C})$ to the hyperplane section bundle of $P^1(\mathbb{C})$ and so $c_1r(H)$ generates $H^2(P^1(\mathbb{C}),\mathbb{Z})$. By the commutativity of the diagram it follows that H generates $H^1(P^n(\mathbb{C}),\mathcal{O}^*)$.

□

We now work towards a classification of holomorphic vector bundles on $P^1(\mathbb{C})$. Suppose $L \in HLB(P^1(\mathbb{C}))$ and $s \in M^*(L)$. Now $\deg(\mathrm{div}(s))$ is independent of s and depends only on L - §5, Chapter 1. We set $\deg(L) = \mathrm{div}(s)$, where s is any non-trivial meromorphic section of L. By Proposition 7.5.7, $L \cong H^d$ for some $d \in \mathbb{Z}$. Since H^d admits a meromorphic section whose divisor had degree d, we see that $\deg(L) = d$. Indeed, $\deg(L) = c_1(L) = c_1(H^d) = d$, where we make the usual identification of $H^2(P^1(\mathbb{C}),\mathbb{Z})$ with $\mathbb{Z}$. Suppose that E is a holomorphic vector bundle on $P^1(\mathbb{C})$ of rank k and let $s \in M^*(E)$ (such sections exist by Theorem 7.5.4). Since the zeroes and poles of s are isolated points, div(s) is a well defined element of $\mathcal{D}(P^1(\mathbb{C}))$ (this would not be the case if E were a holomorphic vector bundle over a complex manifold of dimension greater than 1). For each $z \in P^1(\mathbb{C})$, there exists $f_z \in M_z$, $h_z \in \underline{E}_z$ such that $s_z = f_z h_z$ and $h_z(z) \neq 0$. Define a subset $L^s \subset \underline{E}$ by $L^s_z = h_z \mathcal{O}_z$, $z \in P^1(\mathbb{C})$. It is easily verified that L^s is a locally free subsheaf of $\underline{E}$ of rank 1. Hence L^s is the sheaf of holomorphic sections of a holomorphic line subbundle L^s of E. By the construction $s_z \in M^*(L^s)_z$ for all $z \in P^1(\mathbb{C})$ and so we may regard s as defining a meromorphic section of L^s. Clearly the bundle L^s is uniquely determined by these conditions. Set $d(s) = \deg(L^s)$. We claim that $\max\{d(s): s \in M^*(E)\} < \infty$. Since $L^s \cong \mathcal{O}(d(s))$, we have $\dim_{\mathbb{C}} H^0(P^1(\mathbb{C}),\underline{L}^s) = d(s)+1$ (that is, the dimension of $P^{(d(s))}(\mathbb{C}^2)$). Clearly, $\dim_{\mathbb{C}} H^0(P^1(\mathbb{C}),\underline{E}) \geq \dim_{\mathbb{C}} H^0(P^1(\mathbb{C}),\underline{L}^s)$ for all $s \in M^*(E)$. So if d(s) were unbounded, this would contradict the finiteness of $\dim_{\mathbb{C}} H^0(P^1(\mathbb{C}),\underline{E})$. Set $a_1 = \max_s d(s)$. Choose a holomorphic line subbundle L of E of degree a_1. Thus $\underline{L} \cong \mathcal{O}(a_1)$ and we have the exact sequence

$$0 \to \mathcal{O}(a_1) \to \underline{E} \to \underline{F} \to 0,$$

where $F = E/L$ is a vector bundle of rank $k-1$ on $P^1(\mathbb{C})$.

Theorem 7.5.8 (Grothendieck). Let E be a holomorphic vector bundle on $P^1(\mathbb{C})$ of rank k. Then there exists a unique sequence $a_1 \geq \ldots \geq a_k$ of integers such that

$$\underline{E} \cong \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_k).$$

Proof. We prove by induction on k. Suppose true $k-1$. Then as we showed above there exists a holomorphic line subbundle $L \subset E$ of maximal degree, say a_1, and exact sequence

$$(*)\ldots\ldots \qquad 0 \to \mathcal{O}(a_1) \to \underline{E} \to \underline{F} \to 0,$$

where F is a holomorphic vector bundle of rank $k-1$. By the inductive assumption

$$\underline{F} \cong \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_k),\ a_2 \geq \cdots \geq a_k.$$

We claim $a_1 \geq a_2$. If not, $a_2 \geq a_1 + 1$ and tensoring (*) by $\mathcal{O}(-a_1 - 1)$ we obtain the exact sequence

$$0 \to \mathcal{O}(-1) \to \underline{E}(-a_1 - 1) \to \underline{F}(a_1 - 1) \to 0.$$

Taking the cohomology sequence, together with the vanishing of $H^1(P^1(\mathbb{C}), \mathcal{O}(-1))$, we deduce that

$$H^0(P^1(\mathbb{C}), \underline{E}(-a_1 - 1)) \cong H^0(P^1(\mathbb{C}), \underline{F}(-a_1 - 1)).$$

Now $\underline{F}(-a_1 - 1) \cong \bigoplus_{j=2}^{k} \mathcal{O}(a_j - a_1 - 1)$. Therefore since $a_2 - a_1 - 1 \geq 0$, $H^0(P^1(\mathbb{C}), \underline{F}(-a_1 - 1)) \neq 0$ and so $\underline{E}(-a_1 - 1)$ admits a non-trivial holomorphic section. Hence $E(-a_1 - 1)$ contains a holomorphic line bundle isomorphic to H^p, $p \geq 0$. Consequently, E contains a holomorphic line bundle isomorphic to H^{p+a_1+1} which is of degree $p + a_1 + 1$, contradicting the definition of a_1.

Next we claim that the sequence (*) splits. This is a consequence of a general splitting lemma.

Lemma 7.5.9. Let $0 \to F \xrightarrow{a} G \xrightarrow{b} H \to 0$ be an exact sequence of coherent sheaves on the complex manifold M. Suppose that H is locally free and that $H^1(M, Hom(H,F)) = 0$. Then the sequence splits.

Proof. We refer to the exercises at the end of §1 for the definition of $Hom(F,G)$. See also the exercises at the end of §1, Chapter 6. Since H is locally free the sequence

$$0 \to Hom(H,F) \to Hom(H,G) \to Hom(H,H) \to 0$$

is exact. Taking the cohomology sequence we deduce that

$$H^0(M, Hom(H,G)) \to H^0(M, Hom(H,H)) \to 0$$

is exact. Therefore there exists c: $H \to G$ which is mapped to the identity homomorphism of H. That is, bc = Id. □

Returning to the proof of our theorem we see that

$$Hom(\underline{F},\mathcal{O}(a_1)) = \underline{F}^* \otimes \mathcal{O}(a_1) \cong \bigoplus_{j=2}^{k} \mathcal{O}(a_1 - a_j).$$

Now $a_1 - a_j \geq 0$ and so $H^1(P^1(\mathbb{C}), Hom(\underline{F},\mathcal{O}(a_1)) = 0$. Hence we may apply the splitting lemma to deduce that (*) splits. But then,

$$E \cong \mathcal{O}(a_1) \oplus \bigoplus_{j=2}^{k} \mathcal{O}(a_j).$$

Finally the uniqueness of the sequence $a_1 \geq \dots \geq a_k$ follows by observing that the number of a_i's equal to m is precisely $\dim_{\mathbb{C}} H^0(P^1(\mathbb{C}),E(-m))$. □

Remark. Much is known about the classification of holomorphic vector bundles on compact Riemann surfaces. See Gunning [3], M. Narasimhan [1] and Tjurin [1].

Theorem 7.5.8 does *not* generalise to vector bundles over $P^n(\mathbb{C})$, $n > 1$.

Example 1. $TP^2(\mathbb{C})$ is not a sum of holomorphic line bundles. Suppose the contrary. Then $\underline{TP^2(\mathbb{C})} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$, where we may suppose $a \geq b$. Taking the Euler sequence for $TP^2(\mathbb{C})$, we therefore have the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^3 \to \mathcal{O}(a) \oplus \mathcal{O}(b) \to 0.$$

Suppose $a \geq 2$. Tensoring the Euler sequence by $\mathcal{O}(-2)$ and taking the cohomology sequence of the resulting short exact sequence we find $H^0(P^2(\mathbb{C}),\mathcal{O}(-1)^3) \cong H^0(P^2(\mathbb{C}),\mathcal{O}(a-2) \oplus \mathcal{O}(b-2))$, which cannot be since the first cohomology group is zero, the second non-zero. Therefore, $a,b \leq 1$. Now take the cohomology sequence of the Euler sequence and count dimensions of the zero dimensional groups to derive a contradiction. (The same analysis will show that $TP^n(\mathbb{C})$ is never a direct sum of line bundles for $n \geq 2$).

Suppose that $\ell \subset P^n(\mathbb{C})$ is a line (that is, $\ell \cong P^1(\mathbb{C})$). Grothendieck's theorem, together with a similar analysis to that presented in the example above, shows that

$$\underline{TP}^n(\mathbb{C})|\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}.$$

We say that a holomorphic vector bundle E on $P^n(\mathbb{C})$ of rank is *uniform* of splitting type $(a_1,\dots,a_k)$ if for every complex line $\ell \subset P^n(\mathbb{C})$,

$$\underline{E}|\ell = \bigoplus_{j=1}^{k} \mathcal{O}(a_j).$$

It can be shown that a uniform bundle of rank $k < n$ is a direct sum of line bundles and that if $k = n$ then it is either a direct sum of line bundles or of the form $TP^n(\mathbb{C})(m)$, $TP^n(\mathbb{C})^*(m)$ (see Okonek, Schneider and Spindler [1; pages 70,71]).

However, not every holomorphic vector bundle on $P^n(\mathbb{C})$ is uniform, $n > 1$, and the problem of classification is still open. Substantial progress has recently been made in the classification of a class of rank 2 vector bundles over $P^3(\mathbb{C})$ - the so called *instanton bundles* - which give rise to self dual solutions of the Yang-Mills field equations. An important feature here is the appearance of moduli in the classification. For references, together with an up-to-date survey of the theory of holomorphic vector bundles on projective space, we refer to the book by Okonek, Schneider and Spindler [1].

Exercises.

1. Let E be a holomorphic vector bundle on $P^n(\mathbb{C})$ and $v \in E_z$, $z \in P^n(\mathbb{C})$. Show that there exists $s \in \mathcal{M}^*(E)$ such that $s(z) = v$.

2. Let $M \subset P^n(\mathbb{C})$ be algebraic. Show that Serre's Theorems A and B hold for all coherent sheaves on M (twist with the hyperplane section bundle restricted to M).

3. Verify that
   1. $TP^n(\mathbb{C})$ is not a sum of line bundles, $n > 2$.
   2. $TP^n(\mathbb{C})$ is uniform of splitting type $(2,1,\dots,1)$, $n \geq 1$.

(Hint: It may be useful to recall that $\dim_{\mathbb{C}} P^{(m)}(\mathbb{C}^{n+1}) = \binom{m+n}{n}$).

4. Show that Grothendieck's theorem (Theorem 7.5.8) is equivalent to the following statement about invertible holomorphic matrices: Regarding $P^1(\mathbb{C})$ as the 1-point compactification of $\mathbb{C}$, let $t > 1$ and set $U_1 = \{z \in P^1(\mathbb{C}): |z| < t\}, U_2 = \{z \in P^1(\mathbb{C}): |z| > 1\}$. Suppose $M \in GL(n,A(U_{12}))$. Then there exist $P \in GL(n,A(U_1))$, $Q \in GL(n,A(U_2))$ such that PMQ is a diagonal matrix with diagonal terms z^{a_j}, $a_j \in \mathbb{Z}$.

5. (Koszul complex) Let A be a commutative ring with 1 and for $m \geq 1$, let $E_1,\dots,E_m$ denote the standard A-module basis of A^m (that is $E_1 = (1,0,\dots,0),\dots,E_m = (0,0,\dots,1)$). The A-module $\Lambda^p A^m$ has A-module basis $\{E_J = E_{j_1} \wedge \dots \wedge E_{j_p} : 1 \leq j_1 < \dots < j_p \leq n\}$. Suppose that $f_1,\dots,f_m \in A$, set $f = (f_1,\dots,f_m) \in A^m$ and let I denote the ideal in A generated by $f_1,\dots,f_m$. Show

a) We have the *Koszul complex*

$$0 \to \Lambda^m A^m \xrightarrow{\partial} \Lambda^{m-1} A^m \xrightarrow{\partial} \dots \xrightarrow{\partial} \Lambda^1 A^m \xrightarrow{\partial} A \to A/I \to 0$$

of A-module homomorphisms, where $\partial a = C_f a$. That is, if $a = a_J E_J \in \Lambda^p A^m$ then

$$\partial a = \sum_J a_J \sum_{i=1}^{p} (-1)^{i+1} f_{j_i} E_{j_1} \wedge \dots \wedge \hat{E}_{j_i} \wedge \dots \wedge E_{j_p} .$$

Show also that $\partial(\Lambda^1 A^m) = I$.

b) Let $I_k = (f_1,\dots,f_k)$ denote the ideal in I generated by $f_1,\dots,f_k$. We say that $(f_1,\dots,f_m)$ is a *regular sequence* if f_k is not a zero divisor in A/I_{k-1}, $k = 1,\dots,m$.
Show that if $(f_1,\dots,f_m)$ is a regular sequence, then the Koszul complex is exact (Hint: prove by induction on m. See also Griffiths and Harris [1; pages 688-690]).

Now suppose that $p_j \in \mathbb{C}[z_0,\dots,z_n]$ is homogeneous of degree d_j, $1 \leq j \leq m$, and let $p = (p_1,\dots,p_m)$ denote the corresponding section of $E = \mathcal{O}(d_1) \oplus \dots \oplus \mathcal{O}(d_m)$. Set $X = Z(p) \subset P^n(\mathbb{C})$. Show

c) If $(p_1,\ldots,p_m)$ is a regular sequence in $\mathbb{C}[z_0,\ldots,z_n]$ then the Koszul complex

$$0 \to \Lambda^m \underline{E}^* \xrightarrow{\partial} \Lambda^{m-1}\underline{E}^* \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^1 \underline{E}^* \to \mathcal{O}_{P^n(\mathbb{C})} \to \mathcal{O}_X \to 0$$

is a locally free resolution of $\mathcal{O}_X$.

(See also Exercise 14, §1, Chapter 6. The assumption on $(p_1,\ldots,p_m)$ amounts to saying that X is a *complete intersection*).

6. Let E be a holomorphic vector bundle on the compact Riemann surface M. Given $d \in \mathcal{D}(M)$, we set $E(d) = E \otimes [d]$. Suppose $d, d' \in \mathcal{D}(M)$ and that $d \le d'$. Set $d' - d = \sum_{i=1}^{k} p_i . z_i$, where each $p_i > 0$. Show

a) We have a natural exact sequence

$$0 \to \underline{E}(d) \to \underline{E}(d') \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is the "Manhatten" sheaf whose stalk is zero except at points $x_i \in |d' - d|$ where we have $\mathcal{F}_{x_i} = \mathbb{C}^{p_i q}$, $q = \dim(E)$.

b) If we define $\chi(E) = \dim_{\mathbb{C}} H^0(M,E) - \dim_{\mathbb{C}} H^1(M,E)$ then

$$\chi(E(d)) = q\deg(d) + \chi(E).$$

(Hint: The alternating sum of dimensions in an exact sequence $0 \to L_0 \to L_1 \to \cdots \to L_n \to 0$ of vector spaces is zero).

c) $\dim_{\mathbb{C}} \Omega(E(d)) \ge q\deg(d) + \chi(E)$.

d) E has non-trivial meromorphic sections.

e) In case E is a holomorphic line bundle on M, there exists $d \in \mathcal{D}(M)$ such that $E \cong [d]$ and $c_1(E) = \deg(d)$.

f) $H^1(M, \mathcal{M}^*) = 0$.

7. Let M be a compact Riemann surface. Given $d \in \mathcal{D}(M)$, set $l(d) = \dim_{\mathbb{C}} H^0(M,[d])$; $i(d) = \dim_{\mathbb{C}} H^1(M,[d])$. Show that

$$l(d) - i(d) = \deg(d) + 1 - g,$$

where $g = \dim_{\mathbb{C}} H^1(M,\mathcal{O})$.

(Riemann-Roch formula).

Deduce

a) $l(d) \geq \deg(d) + 1 - g$ (Riemann's inequality).

b) If $\deg(d) \geq g+1$, there exists $m \in M^*(M)$ such that $\operatorname{div}(m) + d \geq 0$.

c) M may be represented as a branched over of $P^1(\mathbb{C})$ with at most $g+1$ sheets. In particular, if $g = 0$, $M = P^1(\mathbb{C})$.

(We remark that g is actually the genus of M. In particular, g is a topological rather than analytic invariant of M. To see that g is the genus of M (that is $\frac{1}{2}\dim_{\mathbb{C}} H^1(M,\mathbb{C})$), we note that Serre-duality implies that $H^1(M,\mathcal{O}) = H^0(M,\Omega^1)$. Now take the cohomology sequence of the short exact sequence $0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{\partial = d} \Omega^1 \to 0$ - see Example 23, §1, Chapter 6).

§6. The Kodaira embedding theorem.

In this section we prove a version of the Kodaira embedding theorem due to Grauert. The main step in the proof is a cohomology vanishing theorem for coherent sheaves on a compact complex manifold admitting a weakly positive vector bundle (for the definition of weak positivity, see §10, Chapter 5).

First some notation. Suppose that E is a holomorphic vector bundle on the complex manifold M. We let $E^{(s)}$ denote the s-fold *symmetric* tensor product of E (this is just the usual tensor product if E is a line bundle). If F is a coherent sheaf on M, we let $F^{(s)}(E)$ denote the E-twisted sheaf $F \otimes_{\mathcal{O}} \underline{E}^{(s)}$.

Theorem 7.6.1. (Grauert [1]). Let E be a weakly positive vector bundle on the compact complex manifold M. Then for any coherent sheaf F on M, there exists $m_0 = m_0(F)$ such that

$$H^p(M,F^{(m)}(E)) = 0, \; p \geq 1, \; m \geq m_0.$$

Proof. Let $\pi: E^* \to M$ denote the dual of E. Then E^* is weakly negative and so there exists an s.L.p. neighbourhood D of the zero section of E^*. Set $\tilde{F} = \pi^*F|D$. Then $\tilde{F}$ is a coherent sheaf on D (Example 6, §1). We shall show that for $N \geq 0$, there exists a canonical linear injection

$$\Phi_N: \sum_{s=0}^{N} H^p(M,F^{(s)}(E)) \to H^p(D,\tilde{F}).$$

Since Grauert's finiteness theorem implies that $H^p(D,\tilde{F})$ is finite dimensional, $p \geq 1$, the existence of such an injection certainly implies that $H^p(M,F^{(s)}(E))$ is zero dimensional for all sufficiently large s, $p \geq 1$.

Let ω denote the zero section of E^* and $i: M \to E^*$ the canonical inclusion of M on ω. Set $\hat{\mathcal{O}} = i^{-1}\mathcal{O}_{E^*}$ and note that $\hat{\mathcal{O}}$ has the natural structure of a sheaf of $\mathcal{O}_M$-modules (not of finite type). For $s \geq 0$, we have a natural $\mathcal{O}_M$-morphism

$$\chi_s: \underline{E}^{(s)} \to \hat{\mathcal{O}}$$

defined by $\chi_s(f_z)(z)(v_z) = f_z(z)(v_z^s)$, $z \in M$, $f_z \in \underline{E}_z^{(s)}$, $v_z \in E_z^*$. That is, $f_z(z) \in E_z^{(s)} \approx (E_z^*)^{*(s)}$ and so defines a homogeneous polynomial map of degree s from E_z^* to $\mathbb{C}$. Evaluate $f_z(z)$ at points of the fibre E_z^*. The map χ_s is obviously injective. Moreover, if we set

$$\chi^N = \bigoplus_{s=0}^{N} \chi_s: \bigoplus_{s=0}^{N} \underline{E}^{(s)} \to \hat{\mathcal{O}}$$

it is clear that χ^N is also injective since the image of each χ_s consists of analytic germs which are homogeneous of degree s in the fibre coordinates. Furthermore, we can define an $\mathcal{O}_M$-morphism $j^N: \hat{\mathcal{O}} \to \bigoplus_{s=0}^{N} \underline{E}^{(s)}$ which is a right inverse for χ^N (that is, $j^N\chi^N = \mathrm{Id}$). To do this suppose $g_z \in \hat{\mathcal{O}}_z$. Then, by Taylor's theorem, we may write $g_z = \sum_{j=0}^{\infty} f_p P_j$, where $f_j \in \mathcal{O}_{M,z}$, $P_j \in \underline{E}_z^{**(s)} \approx \underline{E}_z^{(s)}$. Now define $j^N(g_z) = (f_0P_0,\ldots,f_NP_N)$ (j^Ng_z is the N-jet of g_z along the fibres of E^*). Since i is a biholomorphism, we have induced maps

$$I_N: F \otimes \bigoplus_{s=0}^{N} \underline{E}^{(s)} \longrightarrow \tilde{F}_\omega \ (= (\pi^{-1}F)_\omega \otimes_{\pi^{-1}(\mathcal{O}_M)} \mathcal{O}_\omega)$$

$$J_N: \tilde{F}_\omega \longrightarrow F \otimes \bigoplus_{s=0}^{N} \underline{E}^{(s)}$$

satisfying $J_N I_N = \mathrm{Id}$, and induced maps on cohomology

$$H^p(M, F \otimes \bigoplus_{s=0}^{N} \underline{E}^{(s)}) \underset{J_N^*}{\overset{I_N^*}{\rightleftarrows}} H^p(\omega, \tilde{F}_\omega)$$

(Exercise 14, §3, Chapter 6). In particular, I_N^* is injective, $p, N \geq 0$. Let $r = i^{-1}(\pi|D): D \to \omega$ and $k: \omega \to D$ denote the inclusion. Now rk is the identity map on ω and so we have the commutative diagram

$$\begin{array}{ccc} H^p(\omega, \tilde{F}_\omega) & \xrightarrow{\mathrm{Id}} & H^p(\omega, \tilde{F}_\omega) \\ & {\scriptstyle r^*} \searrow \quad \nearrow {\scriptstyle k^*} & \\ & H^p(D, \tilde{F}) & \end{array}$$

(see Exercise 15, §3, Chapter 6 and note that $k^{-1}\tilde{F} = \tilde{F}_\omega$, $r^{-1}\tilde{F}_\omega = \tilde{F}$). But therefore r^* is injective and so the map

$$\Phi_N = r^* I_N^*: H^p(M, F \otimes \bigoplus_{s=0}^{N} \underline{E}^{(s)}) \to H^p(D, \tilde{F})$$

is injective, $p \geq 0$. □

Suppose that F is a holomorphic line bundle on the compact complex manifold M and that for each $z \in M$, there exists $s \in H^0(M, \underline{F})$ such that $s(z) \neq 0$. Choose a $\mathbb{C}$-basis $s_0, \ldots, s_k$ for $H^0(M, \underline{F})$. Then, as we showed in §9, Chapter 5, $s = (s_0, \ldots, s_k)$ defines a holomorphic map of M in $P^k(\mathbb{C})$. We shall say that F is *ample* (respectively *very ample*) if $H^0(M, \underline{F})$ determines a holomorphic map (respectively embedding) of M in some projective space. We remark that if F is ample (respectively, very ample) then so is F^k, $k \geq 1$.

Theorem 7.6.2. (Kodaira embedding theorem). Let M be a compact complex manifold and suppose that E is a weakly positive vector bundle on M. Then M admits an embedding in projective space.

Proof. (Grauert [1]). We shall prove the theorem in case E is a weakly positive line bundle. The proof in case E is a bundle of rank greater than 1 is similar except that M gets embedded in a Grassmann manifold (which, of course, can be embedded in projective space).

First we show that there exists k_0 such that $H^0(M,\underline{E}^k)$ determines a holomorphic immersion of M in projective space, $k \geq k_0$.

Fix $z \in M$ and let $I_z^2 = I_z I_z$, where I_z is the ideal sheaf of z. The sheaf I_z^2 is coherent and we have the short exact sequence

$$(*)\ldots\qquad 0 \to I_z^2 \to \mathcal{O}_M \xrightarrow{T} J(z) \to 0,$$

where J(z) is the skyscrapper sheaf supported at z with stalk $\mathcal{O}_z/m_z^2$. Now $T(f_z) = (f_z(z), df_z(z))$. That is, $\mathcal{O}_z/m_z^2$ measures the first two terms in the Taylor expansion of f_z. By Theorem 7.5.1, there exists k(z) such that $H^1(M, I_z^2 \otimes \underline{E}^k) = 0$, $k \geq k(z)$. Tensoring (*) with $\underline{E}^k$ and taking the cohomology sequence of the short exact sequence we deduce that

$$H^0(M,\underline{E}^k) \xrightarrow{T^*} H^0(M, J(z) \otimes \underline{E}^k))$$

is surjective, $k \geq k(z)$. Since $H^0(M,J(z) \otimes \underline{E}^k) \cong J(z)$, we deduce that $H^0(M,\underline{E}^k)$ determines a holomorphic embedding of some open neighbourhood U_z of z in projective space, $k \geq k(z)$ (U_z may be chosen independently of $k \geq k(z)$. Indeed, a U_z that works for k(z) will work for $k > k(z)$). Doing this for every $z \in M$ and using the compactness of M we obtain a finite cover $U_1,\ldots,U_n$ of M, corresponding integers $k_1,\ldots,k_n$, such that $H^0(M,\underline{E}^{k_j})$ determines a holomorphic embedding of U_j in projective space. Taking $k_0 = \max_j k_j$, we see that $H^0(M,\underline{E}^k)$ determines a holomorphic immersion of M in projective space, $k \geq k_0$. Set $U = \bigcup_{j=1}^{n} (U_j \times U_j) \subset M \times M$. For $(x,y) \in (M \times M) \setminus U$, let $I_{x,y}$ denote the sheaf of germs of holomorphic functions on M vanishing on $\{x,y\}$. Just as we did above, we find that there exists m(x,y) such that for $m \geq m(x,y)$ the natural map

$$H^0(M,\underline{E}^m) \to H^0(M, (\mathbb{C}(x) \oplus \mathbb{C}(y)) \otimes \underline{E}^m)$$

is onto. Hence there exists a neighbourhood W of (x,y) in $M \times M$ such that if $(x',y') \in W$, there exists $s \in H^0(M,\underline{E}^m)$ with $s(x') \neq s(y')$, $m \geq m(x,y)$. Since $(M \times M) \setminus U$ is compact, we may find an open cover $W_1,\ldots,W_p$ of $(M \times M) \setminus U$ and corresponding integers $m_1,\ldots,m_p$ such that given $(x,y) \in W_j$, there exists $s \in H^0(M,\underline{E}^m)$ with $s(x) \neq s(y)$, $m \geq m_j$. Now let $m_0 = \max\{k_0,m_1,\ldots,m_p\}$. Our construction guarantees that E^m is very ample for $m \geq m_0$. □

Remarks.

1. Kodaira's original proof of Theorem 7.6.2 ia rather different from that of Grauert. Theorem 7.6.1 is replaced by a cohomology vanishing theorem for cohomology with coefficients in a holomorphic vector bundle whose curvature satisfies certain positivity conditions. In the construction of the embedding, Kodaira uses blowing up arguments, in combination with his vanishing theorem, rather than the twisting arguments we used. The reader may consult Kodaira and Morrow [1] and Wells [1] for presentations of Kodaira's original proof. In Chapter 10, we shall prove the cohomology vanishing theorem ("Kodaira's vanishing theorem") referred to above. One important feature of Grauert's proof of the Kodaira embedding theorem is that it generalises to compact analytic spaces admitting a weakly positive bundle. For details we refer to Grauert [1].

2. Since $P^n(\mathbb{C})$ admits a weakly positive line bundle $(H^!)$, Theorem 7.6.2 implies part B of Theorem 7.5.3 of Serre. But part B easily implies part A by the usual cohomology sequence arguments and so we see that Theorem 7.6.1 may be regarded as a generalisation of Theorems A and B of Serre.

3. Chow's theorem implies that the image of M in $P^N(\mathbb{C})$ given by Kodaira's embedding theorem is an algebraic set. We may actually embed M in $P^{2m+1}(\mathbb{C})$, $m = \dim_{\mathbb{C}} M$. We indicate briefly why this is so (for details see Griffiths and Harris [1] and also Hartshorne [1] for the case of curves). Let $\Sigma \subset P^N(\mathbb{C})$ denote the union of all projective lines which are either chords or tangents to M (that is, its image in $P^N(\mathbb{C})$). It may be shown that Σ is an algebraic subset of $P^N(\mathbb{C})$ of dimension $\leq 2m+1$. In particular, $P^N(\mathbb{C}) \setminus \Sigma$ will not be empty provided $N > 2m+1$. Choose any $P^{N-1}(\mathbb{C}) \subset P^N(\mathbb{C})$ and

$P \in P^N(\mathbb{C}) \setminus (\Sigma \cup P^{N-1}(\mathbb{C}))$. Project M into $P^{N-1}(\mathbb{C})$ from P. Clearly the projection is an injective immersion - $P \notin \Sigma$ - and so defines an embedding of M in $P^{N-1}(\mathbb{C})$.

We end with an example.

Example. Let $\Lambda \subset \mathbb{C}^n$ be a lattice and suppose that Λ admits a Riemann form. That is, we shall assume that there exists a positive definite Hermitian form H on $\mathbb{C}^n$ whose imaginary part is integer valued on Λ (see Chapter 4, §4; Chapter 5, §9). We claim that $T = \mathbb{C}^n/\Lambda$ is an abelian variety. We shall prove this by showing that the holomorphic line bundle $L(H,1)$ on T is weakly positive and applying Kodaira's embedding theorem. By Exercise 6, §9, Chapter 5, $\theta(z) = \exp\pi H(z,z)$, $z \in \mathbb{C}^n$, determines a smooth nowhere vanishing section of $L(H,1) \otimes \overline{L(H,1)}$. Define $\eta: \mathbb{C}^n \times \mathbb{C} \to \mathbb{R}$ by $\eta(z,t) = |t|^2\theta(z)$ and observe that η induces a smooth map $\tilde{\eta}: L(H,1)^* \cong L(-H,1) \to \mathbb{R}$ ($\tilde{\eta}$ is the square of the radius function on $L(H,1)^*$ associated to the hermitian metric on $L(H,1)^*$ determined by θ). A straightforward computation shows that $L(\eta)(z,t)$ is positive definite provided that $t \neq 0$. Hence $L(\tilde{\eta})$ is certainly positive definite off the zero section of $L(H,1)^*$. Setting $D = \{v \in L(H,1)^*: \tilde{\eta}(v) < 1\}$, we see that D is an s.L.p. neighbourhood of the zero section of $L(H,1)^*$ and so $L(H,1)^*$ is weakly negative. Hence $L(H,1)$ is weakly positive.

Exercises.

1. Prove Theorem 7.6.2 in case E is of rank greater than 1.

2. This exercise is a continuation of exercises 6, 7 of §5. Let M be a compact Riemann surface. Show that if $z \in M$ then $I_z^p = [\underline{-p.z}]$, $p \geq 0$, where I_z denotes the ideal sheaf of $\{z\}$. Suppose that E is a holomorphic line bundle on M. Prove

   a) If $c_1(E) \geq 2g-1$, then $H^1(M,\underline{E}) = 0$.

   b) If $c_1(E) \geq 2g$, then E is ample (see the proof of Theorem 7.6.2).

   c) If $c_1(E) \geq 2g+1$, then E is very ample.

Deduce that every compact Riemann surface is algebraic.

3. Show that every compact Riemann surface M admits an open covering by two Stein open sets (Hint: Embed M in $P^N(\mathbb{C})$. Consider the intersection of M with two hyperplanes H_1, H_2 chosen so that $M \cap H_1 \cap H_2 = \emptyset$. Show that $\{M \setminus H_i : i = 1,2\}$ is a Stein open cover of M). Deduce that $H^p(M,F) = 0$, $p \geq 2$, for every coherent sheaf F on M (see also Grauert and Remmert [1; page 210]).

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